1. Let \( n \in \mathbb{Z} \). Prove that if \( n^2 \) is even, then \( n \) is even.

2. Let \( x, y \in \mathbb{R} \). Prove that if \( x \in Q \) and \( y \not\in Q \), then \( x+y \not\in Q \).

3. Prove that \( 5^n + 12n + 31 \) is divisible by 16 for all \( n \in \mathbb{N} \) using induction.

4. Use the axioms for a field to prove the following properties. Specify each axiom you are using, and you can also use the result that \( Z \cdot 0 = 0 = 0 \cdot Z \) for all \( Z \).
   a) If \( xy = 0 \), then \( x = 0 \) or \( y = 0 \).
   b) \((-x) \cdot y = -xy\)

5. Let \( T \subseteq \mathbb{R} \), and let \( E = \{ x \in Q : \exists T \} \). Show that \( \text{sup} E = T \).

6. Use the axioms for an ordered field to prove the following properties. Specify each axiom you are using, and you can also use the result that \( Z \cdot 0 = 0 = 0 \cdot Z \) and any results that you have already proved.
   a) If \( x < 0 \) and \( y < 0 \), then \( xy > 0 \).
   b) \( 1 > 0 \) [Hint: you can use Part a].
   c) If \( x > 0 \), then \( x^{-1} > 0 \).

7. Prove the following version of the Archimedean property:
   The natural numbers \( \mathbb{N} \) have no upper bound in \( \mathbb{R} \).

8. a) Prove that if \( x \in \mathbb{R} \) with \( x \geq 0 \), then there is an integer \( n \) with \( n \leq x < n+1 \).
   b) Explain why your proof in Part a) is not valid when \( x < 0 \).