5.7 VOLUMES OF SOLIDS OF REVOLUTION

- Use the Disk Method to find volumes of solids of revolution.
- Use the Washer Method to find volumes of solids of revolution with holes.
- Use solids of revolution to solve real-life problems.

The Disk Method

Another important application of the definite integral is its use in finding the volume of a three-dimensional solid. In this section, you will study a particular type of three-dimensional solid—one whose cross sections are similar. You will begin with solids of revolution. These solids, such as axles, funnels, pills, bottles, and pistons, are used commonly in engineering and manufacturing.

As shown in Figure 5.25, a **solid of revolution** is formed by revolving a plane region about a line. The line is called the **axis of revolution.**

To develop a formula for finding the volume of a solid of revolution, consider a continuous function f that is nonnegative on the interval $[a, b]$. Suppose that the area of the region is approximated by n rectangles, each of width Δx, as shown in Figure 5.26. By revolving the rectangles about the x-axis, you obtain n circular disks, each with a volume of $\pi[f(x_i)]^2\Delta x$. The volume of the solid formed by revolving the region about the x-axis is approximately equal to the sum of the volumes of the n disks. Moreover, by taking the limit as n approaches infinity, you can see that the exact volume is given by a definite integral. This result is called the **Disk Method.**

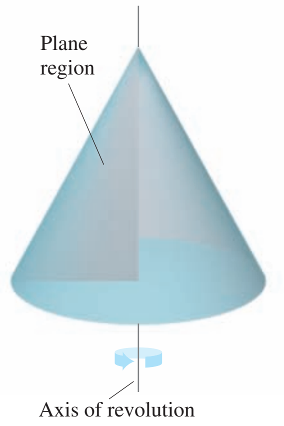
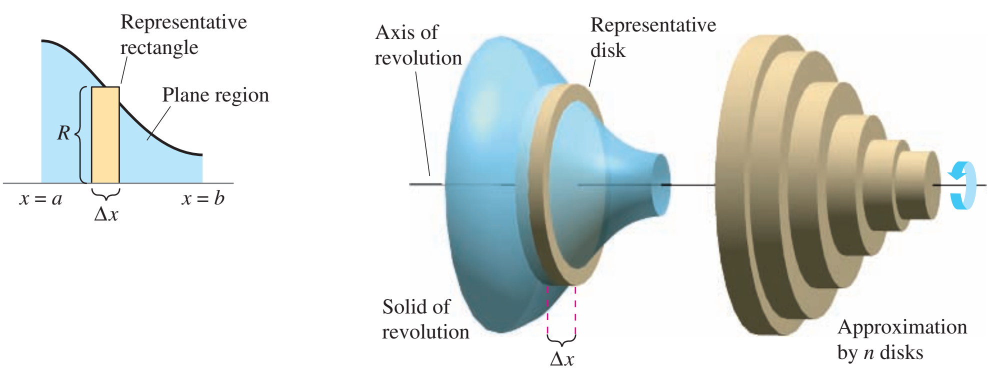


FIGURE 5.25

The Disk Method

The volume of the solid formed by revolving the region bounded by the graph of f and the x-axis $(a \le x \le b)$ about the x-axis is

$$\text{Volume} = \pi\int_a^b [f(x)]^2\,dx.$$



Approximation by n rectangles

FIGURE 5.26

EXAMPLE 1 Finding the Volume of a Solid of Revolution

Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = -x^2 + x$ and the x-axis about the x-axis.

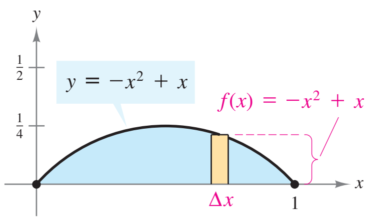
SOLUTION Begin by sketching the region bounded by the graph of f and the x-axis. As shown in Figure 5.27(a), sketch a representative rectangle whose height is $f(x)$ and whose width is Δx. From this rectangle, you can see that the radius of the solid is

$$\text{Radius} = f(x) = -x^2 + x.$$

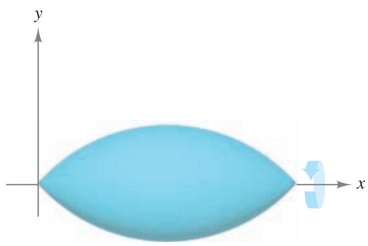
Using the Disk Method, you can find the volume of the solid of revolution.

$$\begin{aligned}
\text{Volume} &= \pi \int_0^1 [f(x)]^2\,dx && \text{Disk Method} \\
&= \pi \int_0^1 (-x^2 + x)^2\,dx && \text{Substitute for } f(x). \\
&= \pi \int_0^1 (x^4 - 2x^3 + x^2)\,dx && \text{Expand integrand.} \\
&= \pi \left[\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3}\right]_0^1 && \text{Find antiderivative.} \\
&= \frac{\pi}{30} && \text{Apply Fundamental Theorem.} \\
&\approx 0.105 && \text{Round to three decimal places.}
\end{aligned}$$

So, the volume of the solid is about 0.105 cubic unit.



(a) Plane region



(b) Solid of revolution

FIGURE 5.27

TECHNOLOGY

Try using the integration capabilities of a graphing utility to verify the solution in Example 1. Consult your user's manual for specific keystrokes.

TRY IT 1

Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = -x^2 + 4$ and the x-axis about the x-axis.

STUDY TIP

In Example 1, the entire problem was solved *without* referring to the three-dimensional sketch given in Figure 5.27(b). In general, to set up the integral for calculating the volume of a solid of revolution, a sketch of the plane region is more useful than a sketch of the solid, because the radius is more readily visualized in the plane region.

The Washer Method

You can extend the Disk Method to find the volume of a solid of revolution with a *hole*. Consider a region that is bounded by the graphs of f and g, as shown in Figure 5.28(a). If the region is revolved about the x-axis, then the volume of the resulting solid can be found by applying the Disk Method to f and g and subtracting the results.

$$\text{Volume} = \pi\int_a^b [f(x)]^2\,dx - \pi\int_a^b [g(x)]^2\,dx$$

Writing this as a single integral produces the **Washer Method.**

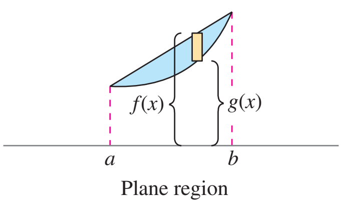
The Washer Method

Let f and g be continuous and nonnegative on the closed interval $[a, b]$, as shown in Figure 5.28(a). If $g(x) \le f(x)$ for all x in the interval, then the volume of the solid formed by revolving the region bounded by the graphs of f and g $(a \le x \le b)$ about the x-axis is

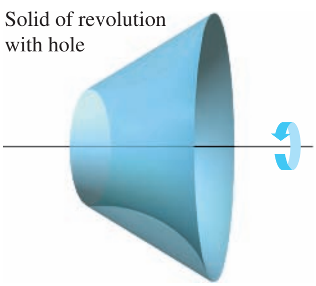
$$\text{Volume} = \pi\int_a^b \{[f(x)]^2 - [g(x)]^2\}\,dx.$$

$f(x)$ is the **outer radius** and $g(x)$ is the **inner radius.**

In Figure 5.28(b), note that the solid of revolution has a hole. Moreover, the radius of the hole is $g(x)$, the inner radius.



(a)



(b)

FIGURE 5.28

EXAMPLE 2 Using the Washer Method

Find the volume of the solid formed by revolving the region bounded by the graphs of

$$f(x) = \sqrt{25 - x^2} \text{ and } g(x) = 3$$

about the x-axis (see Figure 5.29).

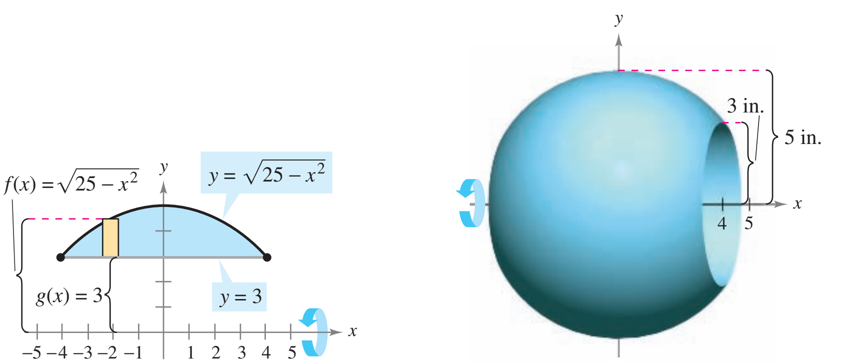
SOLUTION First find the points of intersection of f and g by setting $f(x)$ equal to $g(x)$ and solving for x.

$$\begin{aligned} f(x) &= g(x) && \text{Set } f(x) \text{ equal to } g(x). \\ \sqrt{25 - x^2} &= 3 && \text{Substitute for } f(x) \text{ and } g(x). \\ 25 - x^2 &= 9 && \text{Square each side.} \\ 16 &= x^2 \\ \pm 4 &= x && \text{Solve for } x. \end{aligned}$$

Using $f(x)$ as the outer radius and $g(x)$ as the inner radius, you can find the volume of the solid as shown.

$$\begin{aligned} \text{Volume} &= \pi \int_{-4}^{4} \{[f(x)]^2 - [g(x)]^2\}\, dx && \text{Washer Method} \\ &= \pi \int_{-4}^{4} \left[\left(\sqrt{25 - x^2}\right)^2 - (3)^2\right] dx && \text{Substitute for } f(x) \text{ and } g(x). \\ &= \pi \int_{-4}^{4} (16 - x^2)\, dx && \text{Simplify.} \\ &= \pi \left[16x - \frac{x^3}{3} \right]_{-4}^{4} && \text{Find antiderivative.} \\ &= \frac{256\pi}{3} && \text{Apply Fundamental Theorem.} \\ &\approx 268.08 && \text{Round to two decimal places.} \end{aligned}$$

So, the volume of the solid is about 268.08 cubic inches.



Plane region

(a)

Solid of revolution

(b)

FIGURE 5.29

TRY IT 2

Find the volume of the solid formed by revolving the region bounded by the graphs of $f(x) = 5 - x^2$ and $g(x) = 1$ about the x-axis.

Application

EXAMPLE 3 Finding a Football's Volume

A regulation-size football can be modeled as a solid of revolution formed by revolving the graph of

$$f(x) = -0.0944x^2 + 3.4, \quad -5.5 \le x \le 5.5$$

about the x-axis, as shown in Figure 5.30. Use this model to find the volume of a football. (In the model, x and y are measured in inches.)

SOLUTION To find the volume of the solid of revolution, use the Disk Method.

$$\text{Volume} = \pi \int_{-5.5}^{5.5} [f(x)]^2\,dx \qquad \text{Disk Method}$$

$$= \pi \int_{-5.5}^{5.5} (-0.0944x^2 + 3.4)^2\,dx \qquad \text{Substitute for } f(x).$$

$$\approx 232 \text{ cubic inches} \qquad \text{Volume}$$

So, the volume of the football is about 232 cubic inches.

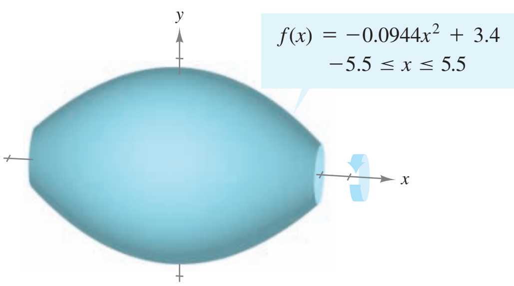


FIGURE 5.30 A football-shaped solid is formed by revolving a parabolic segment about the x-axis.

American football, in its modern form, is a twentieth-century invention. In the 1800s a rough, soccer-like game was played with a "round football." In 1905, at the request of President Theodore Roosevelt, the Intercollegiate Athletic Association (which became the NCAA in 1910) was formed. With the introduction of the forward pass in 1906, the shape of the ball was altered to make it easier to grip.

TRY IT 3

A soup bowl can be modeled as a solid of revolution formed by revolving the graph of

$$f(x) = \sqrt{x} + 1, \quad 0 \le x \le 3$$

about the x-axis. Use this model, where x and y are measured in inches, to find the volume of the soup bowl.

TAKE ANOTHER LOOK

Testing the Reasonableness of an Answer

A football is about 11 inches long and has a diameter of about 7 inches. In Example 3, the volume of a football was approximated to be 232 cubic inches. Explain how you can determine whether this answer is reasonable.

PREREQUISITE REVIEW 5.7

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–6, solve for x.

1. $x^2 = 2x$

2. $-x^2 + 4x = x^2$

3. $x = -x^3 + 5x$

4. $x^2 + 1 = x + 3$

5. $-x + 4 = \sqrt{4x - x^2}$

6. $\sqrt{x - 1} = \frac{1}{2}(x - 1)$

In Exercises 7–10, evaluate the integral.

7. $\displaystyle\int_0^2 2e^{2x}\,dx$

8. $\displaystyle\int_{-1}^3 \frac{2x + 1}{x^2 + x + 2}\,dx$

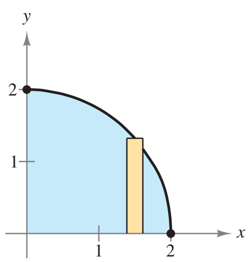
9. $\displaystyle\int_0^2 x\sqrt{x^2 + 1}\,dx$

10. $\displaystyle\int_1^5 \frac{(\ln x)^2}{x}\,dx$

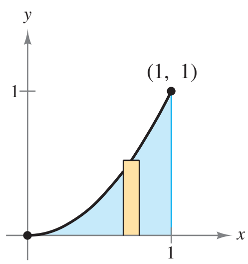
EXERCISES 5.7

In Exercises 1–16, find the volume of the solid formed by revolving the region bounded by the graph(s) of the equation(s) about the x-axis.

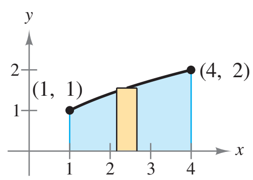
1. $y = \sqrt{4 - x^2}$



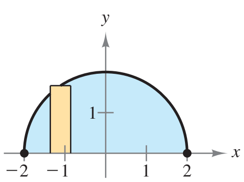
2. $y = x^2$



3. $y = \sqrt{x}$



4. $y = \sqrt{4 - x^2}$



5. $y = 4 - x^2, \quad y = 0$

6. $y = x, \quad y = 0, \quad x = 4$

7. $y = 1 - \frac{1}{4}x^2, \quad y = 0$

8. $y = x^2 + 1, \quad y = 5$

9. $y = -x + 1, \quad y = 0, \quad x = 0$

10. $y = x, \quad y = e^{x-1}, \quad x = 0$

11. $y = \sqrt{x} + 1, \quad y = 0, \quad x = 0, \quad x = 9$

12. $y = \sqrt{x}, \quad y = 0, \quad x = 4$

13. $y = 2x^2, \quad y = 0, \quad x = 2$

14. $y = \dfrac{1}{x}, \quad y = 0, \quad x = 1, \quad x = 3$

15. $y = e^x, \quad y = 0, \quad x = 0, \quad x = 1$

16. $y = x^2, \quad y = 4x - x^2$

In Exercises 17–24, find the volume of the solid formed by revolving the region bounded by the graph(s) of the equation(s) about the y-axis.

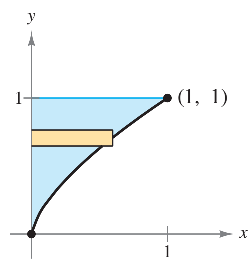
17. $y = x^2, \quad y = 4, \quad 0 \le x \le 2$

18. $y = \sqrt{16 - x^2}, \quad y = 0, \quad 0 \le x \le 4$

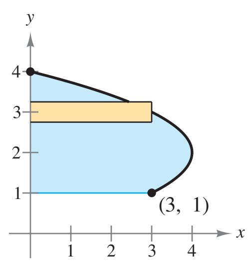
19. $x = 1 - \frac{1}{2}y, \quad x = 0, \quad y = 0$

20. $x = y(y - 1), \quad x = 0$

21. $y = x^{2/3}$



22. $x = -y^2 + 4y$



23. $y = \sqrt{4 - x}, \quad y = 0, \quad x = 0$

24. $y = 4, \quad y = 0, \quad x = 2, \quad x = 0$

25. ***Volume*** The line segment from $(0, 0)$ to $(6, 3)$ is revolved about the x-axis to form a cone. What is the volume of the cone?

26. ***Volume*** The line segment from $(0, 0)$ to $(4, 2)$ is revolved about the y-axis to form a cone. What is the volume of the cone?

27. ***Volume*** Use the Disk Method to verify that the volume of a right circular cone is $\frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height.

28. ***Volume*** Use the Disk Method to verify that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

29. ***Volume*** The right half of the ellipse

$$9x^2 + 25y^2 = 225$$

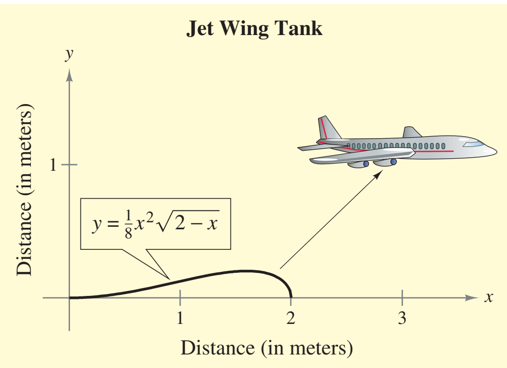
is revolved about the y-axis to form an oblate spheroid (shaped like an M&M candy). Find the volume of the spheroid.

30. ***Volume*** The upper half of the ellipse

$$9x^2 + 16y^2 = 144$$

is revolved about the x-axis to form a prolate spheroid (shaped like a football). Find the volume of the spheroid.

31. ***Volume*** A tank on the wing of a jet airplane is modeled by revolving the region bounded by the graph of $y = \frac{1}{8}x^2\sqrt{2 - x}$ and the x-axis about the x-axis, where x and y are measured in meters (see figure). Find the volume of the tank.



32. ***Volume*** A soup bowl can be modeled as a solid of revolution formed by revolving the graph of

$$y = \sqrt{\frac{x}{2}} + 1, \quad 0 \le x \le 4$$

about the x-axis. Use this model, where x and y are measured in inches, to find the volume of the soup bowl.

33. ***Biology*** A pond is to be stocked with a species of fish. The food supply in 500 cubic feet of pond water can adequately support one fish. The pond is nearly circular, is 20 feet deep at its center, and has a radius of 200 feet. The bottom of the pond can be modeled by

$$y = 20[(0.005x)^2 - 1].$$

(a) How much water is in the pond?

(b) How many fish can the pond support?

34. ***Modeling a Body of Water*** A pond is approximately circular, with a diameter of 400 feet (see figure). Starting at the center, the depth of the water is measured every 25 feet and recorded in the table.

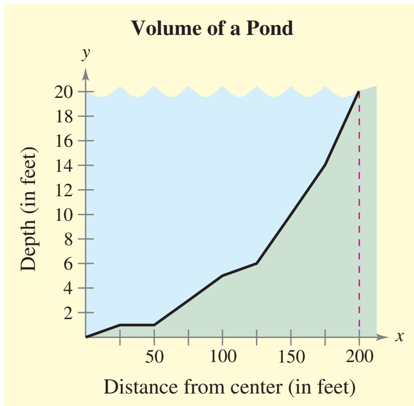
x	0	25	50	75	100
Depth	20	19	19	17	15

x	125	150	175	200
Depth	14	10	6	0

(a) Use a graphing utility to plot the depths and graph the model of the pond's depth, $y = 20 - 0.00045x^2$.

(b) Use the model in part (a) to find the pond's volume.

(c) Use the result of part (b) to approximate the number of gallons of water in the pond ($1 \text{ ft}^3 \approx 7.48$ gal).



In Exercises 35 and 36, use a program similar to the one on page 366 to approximate the volume of a solid generated by revolving the region bounded by the graphs of the equations about the x-axis.

35. $y = \sqrt[3]{x + 1}, \quad y = 0, \quad x = 0, \quad x = 7$

36. $y = \dfrac{10}{x^2 + 1}, \quad y = 0, \quad x = 0, \quad x = 3$