6.3 PARTIAL FRACTIONS AND LOGISTIC GROWTH

- Use partial fractions to find indefinite integrals.
- Use logistic growth functions to model real-life situations.

Partial Fractions

In Sections 6.1 and 6.2, you studied integration by substitution and by parts. In this section you will study a third technique called **partial fractions.** This technique involves the decomposition of a rational function into the sum of two or more simple rational functions. For instance, suppose you know that

$$\frac{x+7}{x^2-x-6} = \frac{2}{x-3} - \frac{1}{x+2}.$$

Knowing the "partial fractions" on the right side would allow you to integrate the left side as shown.

$$\begin{aligned}\int \frac{x+7}{x^2-x-6}\,dx &= \int \left(\frac{2}{x-3} - \frac{1}{x+2}\right) dx \\ &= 2\int \frac{1}{x-3}\,dx - \int \frac{1}{x+2}\,dx \\ &= 2\ln|x-3| - \ln|x+2| + C\end{aligned}$$

To use this method, you must be able to factor the denominator of the original rational function *and* find the partial fraction decomposition of the function.

Partial Fractions

To find the partial fraction decomposition of the *proper* rational function $p(x)/q(x)$, factor $q(x)$ and write an equation that has the form

$$\frac{p(x)}{q(x)} = (\text{sum of partial fractions}).$$

For each *distinct* linear factor $ax + b$, the right side should include a term of the form

$$\frac{A}{ax+b}.$$

For each *repeated* linear factor $(ax + b)^n$, the right side should include n terms of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdot\cdot\cdot + \frac{A_n}{(ax+b)^n}.$$

STUDY TIP

Finding the partial fraction decomposition of a rational function is really a *precalculus* topic. Explain how you could verify that

$$\frac{1}{x-1} + \frac{2}{x+2}$$

is the partial fraction decomposition of

$$\frac{3x}{x^2+x-2}.$$

STUDY TIP

A rational function $p(x)/q(x)$ is *proper* if the degree of the numerator is less than the degree of the denominator.

EXAMPLE 1 Finding a Partial Fraction Decomposition

Write the partial fraction decomposition for

$$\frac{x+7}{x^2-x-6}.$$

SOLUTION Begin by factoring the denominator as $x^2 - x - 6 = (x-3)(x+2)$. Then, write the partial fraction decomposition as

$$\frac{x+7}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}.$$

To solve this equation for A and B, multiply each side of the equation by the least common denominator $(x-3)(x+2)$. This produces the **basic equation** as shown.

$$x + 7 = A(x+2) + B(x-3) \qquad \text{Basic equation}$$

Because this equation is true for all x, you can substitute any convenient values of x into the equation. The x-values that are especially convenient are the ones that make a factor of the least common denominator zero: $x = -2$ and $x = 3$.

Substitute $x = -2$:

$$\begin{aligned}
x + 7 &= A(x+2) + B(x-3) && \text{Write basic equation.}\\
-2 + 7 &= A(-2+2) + B(-2-3) && \text{Substitute } -2 \text{ for } x.\\
5 &= A(0) + B(-5) && \text{Simplify.}\\
-1 &= B && \text{Solve for } B.
\end{aligned}$$

Substitute $x = 3$:

$$\begin{aligned}
x + 7 &= A(x+2) + B(x-3) && \text{Write basic equation.}\\
3 + 7 &= A(3+2) + B(3-3) && \text{Substitute 3 for } x.\\
10 &= A(5) + B(0) && \text{Simplify.}\\
2 &= A && \text{Solve for } A.
\end{aligned}$$

Now that you have solved the basic equation for A and B, you can write the partial fraction decomposition as

$$\frac{x+7}{x^2-x-6} = \frac{2}{x-3} - \frac{1}{x+2}$$

as indicated at the beginning of this section.

ALGEBRA REVIEW

You can check the result in Example 1 by subtracting the partial fractions to obtain the original fraction, as shown in Example 1(b) in the *Chapter 6 Algebra Review*, on page 446.

TRY IT 1

Write the partial fraction decomposition for $\dfrac{x+8}{x^2+7x+12}$.

STUDY TIP

Be sure you see that the substitutions for x in Example 1 are chosen for their convenience in solving for A and B. The value $x = -2$ is selected because it eliminates the term $A(x+2)$, and the value $x = 3$ is chosen because it eliminates the term $B(x-3)$.

TECHNOLOGY

The use of partial fractions depends on the ability to factor the denominator. If this cannot be easily done, then partial fractions should not be used. For instance, consider the integral

$$\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x + 1}\,dx.$$

This integral is only slightly different from that in Example 2, yet it is immensely more difficult to solve. A symbolic integration utility was unable to solve this integral. Of course, if the integral is a definite integral (as is true in many applied problems), then you can use an approximation technique such as the Midpoint Rule.

ALGEBRA REVIEW

You can check the partial fraction decomposition in Example 2 by combining the partial fractions to obtain the original fraction, as shown in Example 1(c) in the *Chapter 6 Algebra Review*, on page 446. Also, for help with the algebra used to simplify the answer, see Example 2(c) on page 447.

EXAMPLE 2 Integrating with Repeated Factors

Find $\displaystyle\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}\,dx.$

SOLUTION Begin by factoring the denominator as $x(x + 1)^2$. Then, write the partial fraction decomposition as

$$\frac{5x^2 + 20x + 6}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}.$$

To solve this equation for A, B, and C, multiply each side of the equation by the least common denominator $x(x + 1)^2$.

$$5x^2 + 20x + 6 = A(x + 1)^2 + Bx(x + 1) + Cx \qquad \text{Basic equation}$$

Now, solve for A and C by substituting $x = -1$ and $x = 0$ into the basic equation.

Substitute $x = -1$:

$$\begin{aligned}
5(-1)^2 + 20(-1) + 6 &= A(-1 + 1)^2 + B(-1)(-1 + 1) + C(-1)\\
-9 &= A(0) + B(0) - C\\
9 &= C \qquad \text{Solve for } C.
\end{aligned}$$

Substitute $x = 0$:

$$\begin{aligned}
5(0)^2 + 20(0) + 6 &= A(0 + 1)^2 + B(0)(0 + 1) + C(0)\\
6 &= A(1) + B(0) + C(0)\\
6 &= A \qquad \text{Solve for } A.
\end{aligned}$$

At this point, you have exhausted the convenient choices for x and have yet to solve for B. When this happens, you can use *any* other x-value along with the known values of A and C.

Substitute $x = 1$, $A = 6$, *and* $C = 9$:

$$\begin{aligned}
5(1)^2 + 20(1) + 6 &= (6)(1 + 1)^2 + B(1)(1 + 1) + (9)(1)\\
31 &= 6(4) + B(2) + 9(1)\\
-1 &= B \qquad \text{Solve for } B.
\end{aligned}$$

Now that you have solved for A, B, and C, you can use the partial fraction decomposition to integrate.

$$\begin{aligned}
\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}\,dx &= \int \left(\frac{6}{x} - \frac{1}{x + 1} + \frac{9}{(x + 1)^2}\right) dx\\
&= 6\ln|x| - \ln|x + 1| + 9\frac{(x + 1)^{-1}}{-1} + C\\
&= \ln\left|\frac{x^6}{x + 1}\right| - \frac{9}{x + 1} + C
\end{aligned}$$

TRY IT 2

Find $\displaystyle\int \frac{3x^2 + 7x + 4}{x^3 + 4x^2 + 4x}\,dx.$

You can use the partial fraction decomposition technique outlined in Examples 1 and 2 only with a *proper* rational function—that is, a rational function whose numerator is of lower degree than its denominator. If the numerator is of equal or greater degree, you must divide first. For instance, the rational function

$$\frac{x^3}{x^2+1}$$

is improper because the degree of the numerator is greater than the degree of the denominator. Before applying partial fractions to this function, you should divide the denominator into the numerator to obtain

$$\frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}.$$

EXAMPLE 3 Integrating an Improper Rational Function

Find $\displaystyle\int \frac{x^5+x-1}{x^4-x^3}\,dx$.

SOLUTION This rational function is improper—its numerator has a degree greater than that of its denominator. So, you should begin by dividing the denominator into the numerator to obtain

$$\frac{x^5+x-1}{x^4-x^3} = x + 1 + \frac{x^3+x-1}{x^4-x^3}.$$

Now, applying partial fraction decomposition produces

$$\frac{x^3+x-1}{x^3(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1}.$$

Multiplying both sides by the least common denominator $x^3(x-1)$ produces the basic equation.

$$x^3+x-1 = Ax^2(x-1) + Bx(x-1) + C(x-1) + Dx^3 \qquad \text{Basic equation}$$

Using techniques similar to those in the first two examples, you can solve for A, B, C, and D to obtain

$$A = 0, \quad B = 0, \quad C = 1, \quad \text{and} \quad D = 1.$$

So, you can integrate as shown.

$$\begin{aligned}\int \frac{x^5+x-1}{x^4-x^3}\,dx &= \int \left(x + 1 + \frac{x^3+x-1}{x^4-x^3}\right) dx \\ &= \int \left(x + 1 + \frac{1}{x^3} + \frac{1}{x-1}\right) dx \\ &= \frac{x^2}{2} + x - \frac{1}{2x^2} + \ln|x-1| + C\end{aligned}$$

ALGEBRA REVIEW

You can check the partial fraction decomposition in Example 3 by combining the partial fractions to obtain the original fraction, as shown in Example 2(a) in the *Chapter 6 Algebra Review*, on page 447.

TRY IT 3

Find $\displaystyle\int \frac{x^4-x^3+2x^2+x+1}{x^3+x^2}$.

Logistic Growth Function

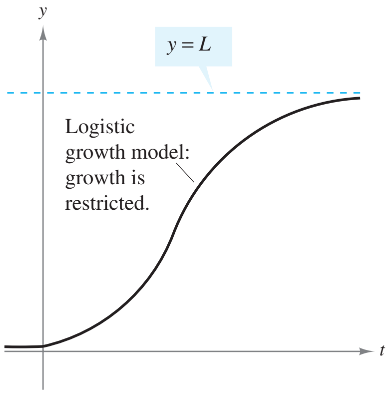


FIGURE 6.6

In Section 4.6, you saw that exponential growth occurs in situations for which the rate of growth is proportional to the quantity present at any given time. That is, if y is the quantity at time t, then

$$\frac{dy}{dt} = ky \qquad \frac{dy}{dt} \text{ is proportional to } y.$$

$$y = Ce^{kt}. \qquad \text{Exponential growth function}$$

Exponential growth is unlimited. As long as C and k are positive, the value of Ce^{kt} can be made arbitrarily large by choosing sufficiently large values of t.

In many real-life situations, however, the growth of a quantity is limited and cannot increase beyond a certain size L, as shown in Figure 6.6. The **logistic growth** model assumes that the rate of growth is proportional to both the quantity y and the difference between the quantity and the limit L. That is

$$\frac{dy}{dt} = ky(L - y). \qquad \frac{dy}{dt} \text{ is proportional to } y \text{ and } (L - y).$$

The solution of this *differential equation* is given in Example 4.

STUDY TIP

The logistic growth model in Example 4 is simplified by assuming that the limit of the quantity y is 1. If the limit were L, then the solution would be

$$y = \frac{L}{1 + be^{-kt}}.$$

In the fourth step of the solution, notice that partial fractions are used to integrate the left side of the equation.

EXAMPLE 4 Deriving the Logistic Growth Function

Solve the equation

$$\frac{dy}{dt} = ky(1 - y).$$

Assume $y > 0$ and $1 - y > 0$.

SOLUTION

$$\frac{dy}{dt} = ky(1 - y) \qquad \text{Write differential equation.}$$

$$\frac{1}{y(1 - y)}\,dy = k\,dt \qquad \text{Write in differential form.}$$

$$\int \frac{1}{y(1 - y)}\,dy = \int k\,dt \qquad \text{Integrate each side.}$$

$$\int \left(\frac{1}{y} + \frac{1}{1 - y}\right) dy = \int k\,dt \qquad \text{Rewrite using partial fractions.}$$

$$\ln y - \ln(1 - y) = kt + C_1 \qquad \text{Find antiderivative.}$$

$$\ln \frac{y}{1 - y} = kt + C_1 \qquad \text{Simplify.}$$

$$\frac{y}{1 - y} = Ce^{kt} \qquad \text{Exponentiate and let } e^{C_1} = C.$$

Solving this equation for y produces

$$y = \frac{1}{1 + be^{-kt}} \qquad \text{Logistic growth function}$$

where $b = 1/C$.

TRY IT 4

Show that if

$$y = \frac{1}{1 + be^{-kt}}, \text{ then}$$

$$\frac{dy}{dt} = ky(1 - y).$$

[*Hint:* First find $ky(1 - y)$ in terms of t, then find dy/dt and show that they are equivalent.]

EXAMPLE 5 Comparing Logistic Growth Functions

Use a graphing utility to investigate the effects of the values of L, b, and k on the graph of

$$y = \frac{L}{1 + be^{-kt}}.\qquad \text{Logistic growth function } (L > 0, b > 0, k > 0)$$

SOLUTION The value of L determines the horizontal asymptote of the graph to the right. In other words, as t increases without bound, the graph approaches a limit of L (see Figure 6.7).

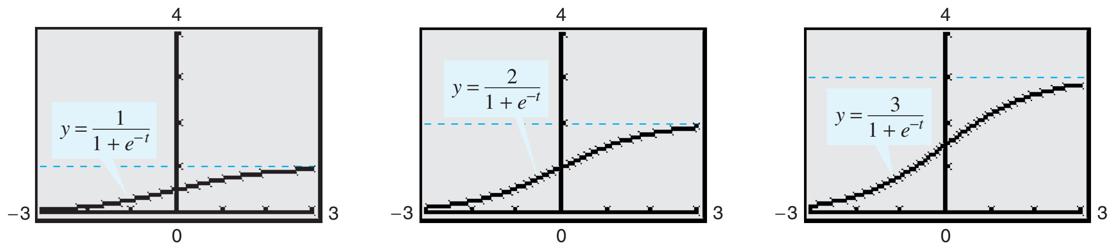


FIGURE 6.7

The value of b determines the point of inflection of the graph. When $b = 1$, the point of inflection occurs when $t = 0$. If $b > 1$, the point of inflection is to the right of the y-axis. If $0 < b < 1$, the point of inflection is to the left of the y-axis (see Figure 6.8).

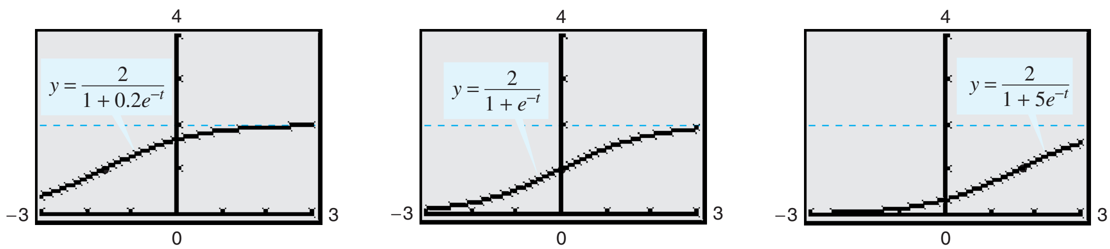


FIGURE 6.8

The value of k determines the rate of growth of the graph. For fixed values of b and L, larger values of k correspond to higher rates of growth (see Figure 6.9).

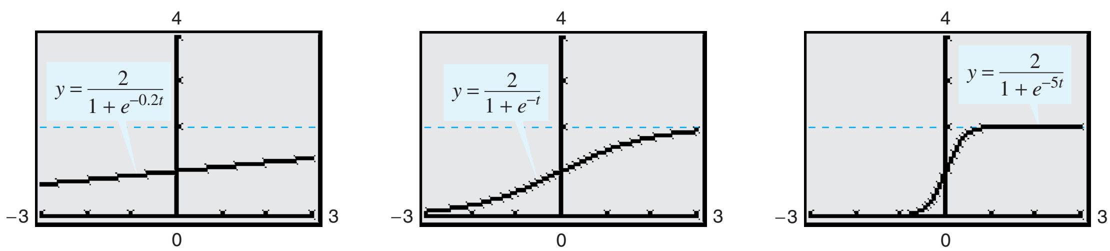


FIGURE 6.9

TRY IT 5

Find the horizontal asymptote of the graph of $y = \dfrac{4}{1 + 5e^{-6t}}$.

EXAMPLE 6 Modeling a Population

The state game commission releases 100 deer into a game preserve. During the first 5 years, the population increases to 432 deer. The commission believes that the population can be modeled by logistic growth with a limit of 2000 deer. Write the logistic growth model for this population. Then use the model to create a table showing the size of the deer population over the next 30 years.

SOLUTION Let y represent the number of deer in year t. Assuming a logistic growth model means that the rate of change in the population is proportional to both y and $(2000 - y)$. That is

$$\frac{dy}{dt} = ky(2000 - y), \qquad 100 \leq y \leq 2000.$$

The solution of this equation is

$$y = \frac{2000}{1 + be^{-kt}}.$$

Using the fact that $y = 100$ when $t = 0$, you can solve for b.

$$100 = \frac{2000}{1 + be^{-k(0)}} \quad \Rightarrow \quad b = 19$$

Then, using the fact that $y = 432$ when $t = 5$, you can solve for k.

$$432 = \frac{2000}{1 + 19e^{-k(5)}} \quad \Rightarrow \quad k \approx 0.33106$$

So, the logistic growth model for the population is

$$y = \frac{2000}{1 + 19e^{-0.33106t}}. \qquad \text{Logistic growth model}$$

The population, in five-year intervals, is shown in the table.

Time, t	0	5	10	15	20	25	30
Population, y	100	432	1181	1766	1951	1990	1998

The American peregrine falcon was removed from the endangered species list in 1999 due to its recovery from 324 nesting pairs in North America in 1975 to 1650 pairs in the United States and Canada. The peregrine was put on the endangered species list in 1970 because of the use of the chemical pesticide DDT. The Fish and Wildlife Service, state wildlife agencies, and many other organizations contributed to the recovery by setting up protective breeding programs among other efforts.

TRY IT 6

Write the logistic growth model for the population of deer in Example 6 if the game preserve could contain a limit of 4000 deer.

TAKE ANOTHER LOOK

Logistic Growth

Analyze the graph of the logistic growth function in Example 6. During which years is the *rate of growth* of the herd increasing? During which years is the *rate of growth* of the herd decreasing? How would these answers change if, instead of a limit of 2000 deer, the game preserve could contain a limit of 3000 deer?

PREREQUISITE REVIEW 6.3

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–8, factor the expression.

1. $x^2 - 16$

2. $x^2 - 25$

3. $x^2 - x - 12$

4. $x^2 + x - 6$

5. $x^3 - x^2 - 2x$

6. $x^3 - 4x^2 + 4x$

7. $x^3 - 4x^2 + 5x - 2$

8. $x^3 - 5x^2 + 7x - 3$

In Exercises 9–14, rewrite the improper rational expression as the sum of a proper rational expression and a polynomial.

9. $\dfrac{x^2 - 2x + 1}{x - 2}$

10. $\dfrac{2x^2 - 4x + 1}{x - 1}$

11. $\dfrac{x^3 - 3x^2 + 2}{x - 2}$

12. $\dfrac{x^3 + 2x - 1}{x + 1}$

13. $\dfrac{x^3 + 4x^2 + 5x + 2}{x^2 - 1}$

14. $\dfrac{x^3 + 3x^2 - 4}{x^2 - 1}$

EXERCISES 6.3

In Exercises 1–12, write the partial fraction decomposition for the expression.

1. $\dfrac{2(x + 20)}{x^2 - 25}$

2. $\dfrac{3x + 11}{x^2 - 2x - 3}$

3. $\dfrac{8x + 3}{x^2 - 3x}$

4. $\dfrac{10x + 3}{x^2 + x}$

5. $\dfrac{4x - 13}{x^2 - 3x - 10}$

6. $\dfrac{7x + 5}{6(2x^2 + 3x + 1)}$

7. $\dfrac{3x^2 - 2x - 5}{x^3 + x^2}$

8. $\dfrac{3x^2 - x + 1}{x(x + 1)^2}$

9. $\dfrac{x + 1}{3(x - 2)^2}$

10. $\dfrac{3x - 4}{(x - 5)^2}$

11. $\dfrac{8x^2 + 15x + 9}{(x + 1)^3}$

12. $\dfrac{6x^2 - 5x}{(x + 2)^3}$

In Exercises 13–32, find the indefinite integral.

13. $\displaystyle\int \frac{1}{x^2 - 1}\,dx$

14. $\displaystyle\int \frac{9}{x^2 - 9}\,dx$

15. $\displaystyle\int \frac{-2}{x^2 - 16}\,dx$

16. $\displaystyle\int \frac{-4}{x^2 - 4}\,dx$

17. $\displaystyle\int \frac{1}{3x^2 - x}\,dx$

18. $\displaystyle\int \frac{3}{x^2 - 3x}\,dx$

19. $\displaystyle\int \frac{1}{2x^2 + x}\,dx$

20. $\displaystyle\int \frac{5}{x^2 + x - 6}\,dx$

21. $\displaystyle\int \frac{3}{x^2 + x - 2}\,dx$

22. $\displaystyle\int \frac{1}{4x^2 - 9}\,dx$

23. $\displaystyle\int \frac{5 - x}{2x^2 + x - 1}\,dx$

24. $\displaystyle\int \frac{x + 1}{x^2 + 4x + 3}\,dx$

25. $\displaystyle\int \frac{x^2 + 12x + 12}{x^3 - 4x}\,dx$

26. $\displaystyle\int \frac{3x^2 - 7x - 2}{x^3 - x}\,dx$

27. $\displaystyle\int \frac{x + 2}{x^2 - 4x}\,dx$

28. $\displaystyle\int \frac{4x^2 + 2x - 1}{x^3 + x^2}\,dx$

29. $\displaystyle\int \frac{4 - 3x}{(x - 1)^2}\,dx$

30. $\displaystyle\int \frac{x^4}{(x - 1)^3}\,dx$

31. $\displaystyle\int \frac{3x^2 + 3x + 1}{x(x^2 + 2x + 1)}\,dx$

32. $\displaystyle\int \frac{3x}{x^2 - 6x + 9}\,dx$

In Exercises 33–40, evaluate the definite integral.

33. $\displaystyle\int_4^5 \frac{1}{9 - x^2}\,dx$

34. $\displaystyle\int_0^1 \frac{3}{2x^2 + 5x + 2}\,dx$

35. $\displaystyle\int_1^5 \frac{x - 1}{x^2(x + 1)}\,dx$

36. $\displaystyle\int_0^1 \frac{x^2 - x}{x^2 + x + 1}\,dx$

37. $\displaystyle\int_0^1 \frac{x^3}{x^2 - 2}\,dx$

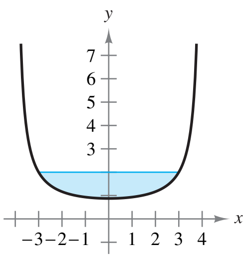
38. $\displaystyle\int_0^1 \frac{x^3 - 1}{x^2 - 4}\,dx$

39. $\displaystyle\int_1^2 \frac{x^3 - 4x^2 - 3x + 3}{x^2 - 3x}\,dx$

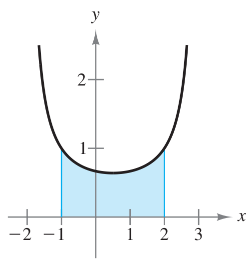
40. $\displaystyle\int_2^4 \frac{x^4 - 4}{x^2 - 1}\,dx$

In Exercises 41–44, find the area of the shaded region.

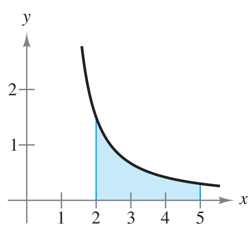
41. $y = \dfrac{14}{16 - x^2}$



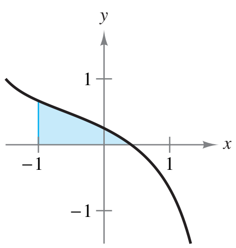
42. $y = \dfrac{-4}{x^2 - x - 6}$



43. $y = \dfrac{x + 1}{x^2 - x}$



44. $y = \dfrac{x^2 + 2x - 1}{x^2 - 4}$



In Exercises 45–48, write the partial fraction decomposition for the rational expression. Check your result algebraically. Then assign a value to the constant *a* and use a graphing utility to check the result graphically.

45. $\dfrac{1}{a^2 - x^2}$

46. $\dfrac{1}{x(x + a)}$

47. $\dfrac{1}{x(a - x)}$

48. $\dfrac{1}{(x + 1)(a - x)}$

In Exercises 49–52, use a graphing utility to graph the function. Then find the volume of the solid generated by revolving the region bounded by the graphs of the given equations about the *x*-axis by using the integration capabilities of a graphing utility and by integrating by hand using partial fraction decomposition.

49. $y = \dfrac{10}{x(x + 10)},\ y = 0,\ x = 1,\ x = 5$

50. $y = \dfrac{-4}{(x + 1)(x - 4)},\ y = 0,\ x = 0,\ x = 3$

51. $y = \dfrac{2}{x^2 - 4},\ x = 1,\ x = -1,\ y = 0$

52. $y = \dfrac{25x}{x^2 + x - 6},\ x = -2,\ x = 0,\ y = 0$

53. ***Biology*** A conservation organization releases 100 animals of an endangered species into a game preserve. The organization believes that the preserve has a capacity of 1000 animals and that the herd will grow according to a logistic growth model. That is, the size y of the herd will follow the equation

$$\int \frac{1}{y(1000 - y)}\,dy = \int k\,dt$$

where t is measured in years. Find this logistic curve. (To solve for the constant of integration C and the proportionality constant k, assume $y = 100$ when $t = 0$ and $y = 134$ when $t = 2$.) Use a graphing utility to graph your solution.

54. ***Health: Epidemic*** A single infected individual enters a community of 500 individuals susceptible to the disease. The disease spreads at a rate proportional to the product of the total number infected and the number of susceptible individuals not yet infected. A model for the time it takes for the disease to spread to x individuals is

$$t = 5010 \int \frac{1}{(x + 1)(500 - x)}\,dx$$

where t is the time in hours.

(a) Find the time it takes for 75% of the population to become infected (when $t = 0$, $x = 1$).

(b) Find the number of people infected after 100 hours.

55. ***Marketing*** After test-marketing a new menu item, a fast-food restaurant predicts that sales of the new item will grow according to the model

$$\frac{dS}{dt} = \frac{2t}{(t + 4)^2}$$

where t is the time in weeks and S is the sales (in thousands of dollars). Find the sales of the menu item at 10 weeks.

56. ***Biology*** One gram of a bacterial culture is present at time $t = 0$, and 10 grams is the upper limit of the culture's weight. The time required for the culture to grow to y grams is modeled by

$$kt = \int \frac{1}{y(10 - y)}\,dy$$

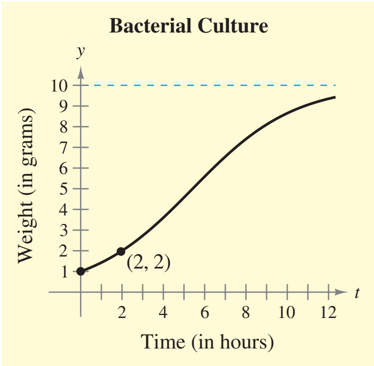
where y is the weight of the culture (in grams) and t is the time in hours.

(a) Verify that the weight of the culture at time t is modeled by

$$y = \frac{10}{1 + 9e^{-10kt}}.$$

Use the fact that $y = 1$ when $t = 0$.

(b) Use the graph to determine the constant k.



57. ***Revenue*** The revenue R (in millions of dollars per year) for Symantec Corporation from 1995 through 2003 can be modeled by

$$R = \frac{410t^2 + 28{,}490t + 28{,}080}{-6t^2 + 94t + 100}$$

where $t = 5$ corrresponds to 1995. Find the total revenue from 1995 through 2003. Then find the average revenue during this time period. *(Source: Symantec Corporation)*

58. ***Medicine*** On a college campus, 50 students return from semester break with a contagious flu virus. The virus has a history of spreading at a rate of

$$\frac{dN}{dt} = \frac{100e^{-0.1t}}{(1 + 4e^{-0.1t})^2}$$

where N is the number of students infected after t days.

(a) Find the model giving the number of students infected with the virus in terms of the number of days since returning from semester break.

(b) If nothing is done to stop the virus from spreading, will the virus spread to infect half the student population of 1000 students? Explain your answer.

59. ***Biology*** A conservation organization releases 100 animals of an endangered species into a game preserve. The organization believes the population of the species will increase at a rate of

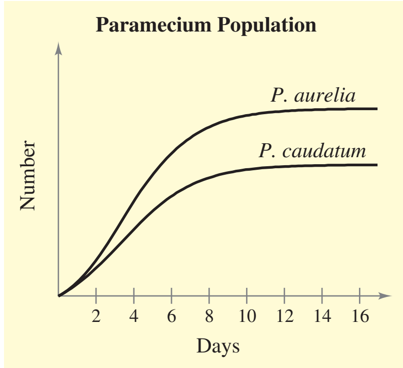
$$\frac{dN}{dt} = \frac{125e^{-0.125t}}{(1 + 9e^{-0.125t})^2}$$

where N is the population and t is the time in months.

(a) Use the fact that $N = 100$ when $t = 0$ to find the population after 2 years.

(b) Find the limiting size of the population as time increases without bound.

60. ***Biology: Population Growth*** The graph shows the logistic growth curves for two species of the single-celled *Paramecium* in a laboratory culture. During which time intervals is the rate of growth of each species increasing? During which time intervals is the rate of growth of each species decreasing? Which species has a higher limiting population under these conditions? *(Source: Adapted from Levine/Miller,* Biology: Discovering Life, *Second Edition)*



BUSINESS CAPSULE

Courtesy of Susie Wang/Aqua Dessa

While a math communications major at the University of California at Berkeley, Susie Wang began researching the idea of selling natural skin-care products. She used \$10,000 to start her company, Aqua Dessa, and uses word-of-mouth as an advertising tactic. Aqua Dessa products are used and sold at spas and exclusive cosmetics counters throughout the United States.

61. ***Research Project*** Use your school's library, the Internet, or some other reference source to research the opportunity cost of attending graduate school for 2 years to receive a Masters of Business Administration (MBA) degree rather than working for 2 years with a bachelor's degree. Write a short paper describing these costs.