1. Prove that \( a^n \geq n^a \) for all integers \( n \geq 4 \).

   **Proof:**
   If \( n = 4 \), this is true since \( 4^4 = 16 > 4^1 \).

   Assume that \( a^n \geq n^a \) for some integer \( n \geq 4 \).

   Then \( 2^{n+1} = 2(2^n) \geq 2n^4 \), and

   \[
   2n^4 = n^4 + 4n = n^4 + 2n + 2n > n^4 + 2n + 1 = (n+1)^2 \text{ if } n \geq 4,
   \]

   so \( 2^{n+1} \geq (n+1)^2 \).

   Therefore, \( a^n \geq n^a \) for all integers \( n \geq 4 \).

2. Use that \( n^2 - 2n = n(n-1) \geq 8 \) if \( n \geq 4 \), so \( 2n^2 \geq n^2 + 8 > n^2 + 2n + 1 = (n+1)^2 \).

   Then \( n \geq 4 \Rightarrow \frac{1}{n} \leq \frac{1}{4} \Rightarrow \frac{(n+1)^2}{n^2} = \left(1 + \frac{1}{n}\right)^2 \leq \left(\frac{5}{4}\right)^2 = \frac{25}{16} < 2 \Rightarrow (n+1)^2 \leq 2n^2 \).

3. Let \( U = \sqrt[4]{3 - \sqrt{5}} \). Prove that if \( U^2 \) is irrational, then \( U \) is irrational.

   **Proof:** (of the contrapositive)

   If \( U \) is rational, then \( U^2 \) is rational (since \( x, y \in \mathbb{Q} \Rightarrow xy \in \mathbb{Q} \)).

4. Prove that \( T = \sqrt[4]{3 - \sqrt{5}} \) is irrational.

   **Proof:** (by contradiction)

   Assume instead that \( T \) is rational, so \( T^4 = 3 - \sqrt{5} \in \mathbb{Q} \) and

   therefore \( \sqrt{5} = 3 - T^4 \in \mathbb{Q} \). This gives a contradiction (since \( 5 \) is not a perfect square), so \( T \) must be irrational.

5. Prove that \( \lim_{n \to \infty} \frac{3n+1}{n+2} = 3 \).

   **Proof:**

   \[
   \left| \frac{3n+1}{n+2} - 3 \right| = \left| \frac{5}{n+2} \right| < \epsilon \iff \frac{5}{n+2} < \frac{1}{\epsilon} \iff n+2 > \frac{5}{\epsilon} \iff n > \frac{5}{\epsilon} - 2
   \]

6. Let \( \epsilon > 0 \) be given, and let \( N \in \mathbb{N} \) with \( N > \frac{5}{\epsilon} - 2 \).

   If \( n \geq N \), then \( n > \frac{5}{\epsilon} - 2 \Rightarrow n+2 > \frac{5}{\epsilon} \Rightarrow \frac{1}{n+2} < \frac{\epsilon}{5} \Rightarrow \frac{5}{n+2} < \epsilon \)

   \[
   \Rightarrow \left| \frac{3n+1}{n+2} - 3 \right| = \left| \frac{5}{n+2} \right| = \frac{5}{n+2} < \epsilon.
   \]

7. Let \( \epsilon > 0 \) be given, and let \( N \in \mathbb{N} \) with \( N > \frac{5}{\epsilon} \).

   If \( n \geq N \), then \( n > \frac{5}{\epsilon} \Rightarrow \frac{1}{n} < \frac{\epsilon}{5} \Rightarrow \frac{5}{n} < \epsilon \)

   \[
   \Rightarrow \left| \frac{3n+1}{n+2} - 3 \right| = \left| \frac{5}{n+2} \right| = \frac{5}{n+2} < \epsilon.
   \]
5. Prove that \( \mathbb{Q} \) is dense in \( \mathbb{R} \).
   
   \( \frac{PF}{1.} \) Let \((x, y)\) be an open interval. Since \( y - x > 0 \) and \( 1 > 0 \), by the Archimedean Property there is an \( n \in \mathbb{N} \) with \( n(y - x) > 1 \). Then \( ny - nx > 1 \), so there is an integer \( m \) in \((nx, ny)\). Since \( nx < m < ny \), \( x < \frac{m}{n} < y \) and therefore \( \frac{m}{n} \) is in \((x, y)\) with \( \frac{m}{n} \in \mathbb{Q} \). Thus \( \mathbb{Q} \) is dense in \( \mathbb{R} \).

6. Prove that if \( a_n \leq s_n \leq c_n \) for all \( n \in \mathbb{N} \) and \( \lim n \rightarrow \infty a_n = l \) and \( \lim n \rightarrow \infty c_n = l \), then \( \lim n \rightarrow \infty s_n = l \).
   
   \( \frac{PF}{1.} \) Let \( \varepsilon > 0 \) be given.
   1) Since \( \lim n \rightarrow \infty a_n = l \), there is an \( N_1 \in \mathbb{N} \) such that \( n \geq N_1 \Rightarrow |a_n - l| < \varepsilon \).
   2) Since \( \lim n \rightarrow \infty c_n = l \), there is an \( N_2 \in \mathbb{N} \) such that \( n \geq N_2 \Rightarrow |c_n - l| < \varepsilon \).
   
   Let \( N = \max\{N_1, N_2\} \).
   
   If \( n \geq N \), then \( |s_n - l| \leq |s_n - a_n| + |a_n - l| + |c_n - l| + |l - c_n| \leq 2\varepsilon + \varepsilon + \varepsilon = 4\varepsilon \), so \( |s_n - l| < \varepsilon \).
   And therefore \( \lim n \rightarrow \infty s_n = l \).

7. Prove that the natural numbers \( \mathbb{N} \) have no upper bound in \( \mathbb{R} \).
   
   \( \frac{PF}{1.} \) (By contradiction)
   
   Suppose instead that \( \mathbb{N} \) is bounded above in \( \mathbb{R} \).
   
   Then by the completeness axiom, \( s = \sup \mathbb{N} \) exists.
   
   Since \( s - 1 < s \), \( s - 1 \) is not an upper bound for \( \mathbb{N} \).
   
   So \( s - 1 \leq m \) for some \( m \in \mathbb{N} \),
   
   Therefore \( s < m + 1 \) with \( m \in \mathbb{N} \), and this contradicts the fact that \( s \) is an upper bound for \( \mathbb{N} \).
   
   Therefore \( \mathbb{N} \) has no upper bound in \( \mathbb{R} \).

8. a) Let \( T = \{ r \in \mathbb{Q} : r > 0 \text{ and } x < r^2 < 3 \} \), for example.
   
   b) Let \( a = 5 - \sqrt{2} \) and \( b = 6 + \sqrt{2} \), for example.
   
   c) Let \( x_n = \sqrt{n} + \frac{1}{n} \) or \( x_n = \sqrt{1 + \frac{1}{n}} \), for example.
If \( T \) is a nonempty bounded subset of \( \mathbb{R} \) and \( a < 0 \), show that \( \inf(aT) = a \sup(T) \).

**PF**

1) Let \( s = \sup(T) \), so \( t \leq s \) for all \( t \in T \).

   Then \( aT \geq as \) for all \( t \in T \), so \( as \) is a lower bound for \( aT \).

2) Let \( q \) be any lower bound for \( aT \), so \( q \leq aT \) for all \( t \in T \).

   Then \( \frac{q}{a} \geq t \) for all \( t \in T \), so \( \frac{q}{a} \) is an upper bound for \( T \).

   And therefore \( s \leq \frac{q}{a} \).

Then \( a\cdot s \geq q \cdot \frac{1}{a} \), so \( a \cdot (\sup(T)) = as = \inf(aT) \).