Problem Sheet 6

1) Let \( s_n = \left(1 + \frac{1}{n}\right)^n \) for all \( n \in \mathbb{N} \). Show that \( \lim_{n \to \infty} s_n = e \) as follows:

You saw in discussion class that \((s_n)\) converges, so let \( \lim_{n \to \infty} s_n = s \).

If \( t_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \) for all \( n \in \mathbb{N} \), we showed in class that \((t_n)\) converges, and we defined \( \lim_{n \to \infty} t_n = e \).

a) Use problem 1b on Discussion Sheet 4 to show that \( s \leq e \).

b) If \( 1 \leq m \leq n \), use the Binomial Theorem to show that

\[
s_n \geq 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{(1-\frac{1}{n})(1-\frac{2}{n})}{3!} + \cdots + \frac{(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{m-1}{n})}{m!}.
\]

c) Use part b) to show that for any \( m \in \mathbb{N} \), \( s \geq t_m \).

d) Use part c) to show that \( s \geq e \).

From parts a) and d), we can conclude that \( \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e \)

2) Let \( s_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \) for all \( n \in \mathbb{N} \).

Use the Monotone Convergence Theorem and the fact that

\[
\frac{1}{m^2} \leq \frac{1}{m(m-1)} = \frac{1}{m-1} - \frac{1}{m} \text{ for } m \geq 2 \text{ to show that } (s_n) \text{ converges}.
\]

3) Let \( s_n = \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \cdots + (k(n))\frac{1}{n!} \), where \( k(n) = \frac{1}{3}(4\cos((2n-3)\frac{\pi}{3}) + 1) \).

a) Show that \( |s_{n+1} - s_n| \leq \frac{1}{2^n} \) for all \( n \in \mathbb{N} \).

b) Use a problem in Sec. 10 to conclude that \((s_n)\) is a Cauchy sequence.