1) Let $s_n = \left(1 + \frac{1}{n}\right)^n$ for all $n \in \mathbb{N}$. Show that $(s_n)$ converges by the MCT using the following steps:

a) Show that $(s_n)$ is increasing. [Hint: Show that $\frac{s_{n+1}}{s_n} \geq 1$ for all $n \in \mathbb{N}$ using Bernoulli’s inequality, or show that $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$ using the Binomial Theorem.]

b) Let $t_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$ for all $n \in \mathbb{N}$.

Show that $(s_n)$ is bounded above by showing that $s_n \leq t_n$ for all $n \in \mathbb{N}$ using the Binomial Theorem, and using the fact that $t_n < 3$ for all $n \in \mathbb{N}$ as shown in class.

2) Let $s_n = \left(1 + \frac{1}{n}\right)^n$ for all $n \in \mathbb{N}$. Show that $\lim_{n \to \infty} s_n = e$ as follows:

We know that $(s_n)$ converges by problem 1, so let $\lim_{n \to \infty} s_n = s$.

If $t_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$ for all $n \in \mathbb{N}$,

we showed in class that $(t_n)$ converges, and we defined $\lim_{n \to \infty} t_n = e$.

a) Use problem 1b) to show that $s \leq e$.

b) If $1 \leq m \leq n$, use the Binomial Theorem to show that

$$s_n \geq 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{(1 - \frac{1}{n})(1 - \frac{2}{n})}{3!} + \cdots + \frac{(1 - \frac{1}{n})(1 - \frac{2}{n}) \cdots (1 - \frac{m-1}{n})}{m!}.$$

c) Use part b) to show that for any $m \in \mathbb{N}$, $s \geq t_m$.

d) Use part c) to show that $s \geq e$.

From parts a) and d), we can conclude that $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$. 

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3) Let $s_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$ for all $n \in \mathbb{N}$.

Use the Monotone Convergence Theorem and the fact that

$$\frac{1}{m^2} \leq \frac{1}{m(m-1)} = \frac{1}{m-1} - \frac{1}{m} \quad \text{for } m \geq 2$$

to show that $(s_n)$ converges.

4) Let $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}$ for all $n \in \mathbb{N}$.

a) Show that $(a_n)$ is increasing by showing that $a_{n+1} - a_n \geq 0$ for all $n \in \mathbb{N}$.

b) Show that $(a_n)$ is bounded above.

c) We can conclude from parts a) and b) that $(a_n)$ converges by the Monotone Convergence Theorem, so find $\lim_{n \to \infty} a_n$ by interpreting this as a limit of a Riemann sum.

5) Let $(s_n)$ be a sequence with $\lim_{n \to \infty} s_n = s$.

Prove that if $(t_n)$ is defined by $t_n = \frac{1}{n}(s_1 + \cdots + s_n)$, then $\lim_{n \to \infty} t_n = s$.

[Hint: Use that $|t_n - s| = \left| \frac{1}{n}(s_1 + \cdots + s_n) - s \right| = \frac{1}{n} |(s_1 - s) + \cdots + (s_n - s)|$.]