1) Show that if \( \sum_{n=1}^{\infty} a_n \) diverges, then \( \sum_{n=1}^{\infty} c a_n \) diverges for any \( c \neq 0 \).

2) Show that if \( \sum_{n=1}^{\infty} a_n \) diverges and \( \sum_{n=1}^{\infty} b_n \) converges, then \( \sum_{n=1}^{\infty} (a_n - b_n) \) diverges.

3) Let \( \sum_{n=1}^{\infty} a_n \) be a series, and let \( \sum_{n=1}^{\infty} b_n \) be a series obtained from \( \sum_{n=1}^{\infty} a_n \) by grouping (inserting parentheses around groups of finitely many terms of \( \sum_{n=1}^{\infty} a_n \)).

Show that if \( \sum_{n=1}^{\infty} a_n \) converges, then \( \sum_{n=1}^{\infty} b_n \) converges and has the same sum.

4) Let \( \sum_{n=1}^{\infty} a_n \) be a series and let \( m \) be in \( \mathbb{N} \).

Show that \( \sum_{n=1}^{\infty} a_n \) converges iff \( \sum_{n=m+1}^{\infty} a_n \) converges, and that \( \sum_{n=1}^{\infty} a_n = S_m + \sum_{n=m+1}^{\infty} a_n \) when both series converge.

5) Let \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) be series with \( a_n = b_n \) for \( n \geq N \), for some \( N \) in \( \mathbb{N} \).

Prove that \( \sum_{n=1}^{\infty} a_n \) converges iff \( \sum_{n=1}^{\infty} b_n \) converges.

6) Let \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) be two convergent series.

If \( a_n \leq b_n \) for all \( n \) in \( \mathbb{N} \), and \( a_m < b_m \) for some \( m \) in \( \mathbb{N} \), show that \( \sum_{n=1}^{\infty} a_n < \sum_{n=1}^{\infty} b_n \).

7) Use the following steps to show that the harmonic series diverges:

a) Show that \( S_{2m} \geq 1 + \frac{m}{2} \) for all \( m \) in \( \mathbb{N} \) using induction.

b) Use part a) to show that \( \lim_{n \to \infty} S_n = \infty \).

8) Identify the results on this sheet which are used in the following proof that the harmonic series diverges:

Assume instead that the harmonic series converges, with \( \sum_{n=1}^{\infty} \frac{1}{n} = S \).

Then \( S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{8} + \cdots = \left( \frac{1}{2} + \frac{1}{2} \right) + \left( \frac{1}{4} + \frac{1}{4} \right) + \left( \frac{1}{6} + \frac{1}{6} \right) + \left( \frac{1}{8} + \frac{1}{8} \right) + \cdots = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = S \), which gives a contradiction.

9) Let \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) be series with all terms nonnegative.

Prove that if \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) both converge, then \( \sum_{n=1}^{\infty} a_n b_n \) converges.