(a) $\sqrt{2}$ is irrational, since 2 is not a perfect square.

(b) $\sqrt{3}$ is irrational, since 3 is not a perfect cube.

(c) $\sqrt{13}$ is irrational, since 13 is not a perfect fourth power.

(Using the corollary to the Rational Roots Theorem, Cor. 2.3).

3. Show that $\sqrt{2} + \sqrt{3}$ is irrational.

**Proof (by contradiction)**

Suppose instead that $x = \sqrt{2} + \sqrt{3}$ is rational.

Then $x^2 = (\sqrt{2} + \sqrt{3})^2 = 5 + 2\sqrt{6}$ is rational, so $\sqrt{6}$ is rational.

This gives a contradiction, since $\sqrt{6}$ is irrational.

Therefore, $\sqrt{2} + \sqrt{3}$ is irrational.

4. Show that $\sqrt{5} - \sqrt{3}$ is irrational.

**Proof (by contradiction)**

Suppose instead that $x = \sqrt{5} - \sqrt{3}$ is rational.

Then $x^2 = 5 - 2\sqrt{15} + 3 = 8 - 2\sqrt{15}$ is rational, so $\sqrt{15}$ is rational.

However, $\sqrt{15}$ is irrational, since 15 is not a perfect square.

So this gives a contradiction, and hence $\sqrt{5} - \sqrt{3}$ is irrational.

5. Proof

Let $x = \sqrt{5} - \sqrt{3}$, so $x^2 = 5 - 2\sqrt{15} + 3 = 8 - 2\sqrt{15}$.

Then $(\sqrt{5} - \sqrt{3})^2 = 3$, so $5 - 2\sqrt{15} + 3 = 3$.

The possible rational roots are $\pm 1, \pm 3, \pm 11, \pm 33$.

Since $\sqrt{15}$ is irrational, $\sqrt{5} - \sqrt{3}$ is irrational.

Therefore, $\sqrt{5} - \sqrt{3}$ is irrational.
3.3 - A) \((a)(b) = ab\) for all \(a, b\)

- **Proof**
  - Using Part 3, \((-a)(-b) = -((a(-b)) = -(-ab) = ab\) (by Cor. 1)

b) if \(a = bc\) and \(c \neq 0\), then \(a = b\).

- **Proof**
  - If \(ac = bc\) and \(c \neq 0\), then \((ac)c^{-1} = (bc)c^{-1}\) (by M1) so
  
  \[a(c^{-1}) = b(c^{-1})\] (by M1).
  
  Then \(a_{c} = b_{c}\) (by M2), so \(a = b\) (by M3).

3.4 - A) \(a < b\)

- **Proof**
  - By Part 4, \(a < b \leq a \leq b\) (by M3); so \(a < b\).

b) if \(a < a \leq b < b^{-1} < a^{-1}\).

- **Proof**
  - By Part 5, \(a^{-1} > 0\) and \(b^{-1} > 0\), since \(a < b\), \(a^{-1} < a^{-1}b\) (by OS) so
  
  \[1 < a^{-1}b\] (by M1). Then \(1(b^{-1}) < (a^{-1})(b^{-1})\) (by OS), so
  
  \[b^{-1} < a^{-1}(b^{-1})\] (by M3 and M1) and \(b^{-1} < a^{-1} < a^{-1}(b^{-1})\) (by M4 and M5),
  
  therefore \(c < b^{-1} < a^{-1}\).

P.5.2 - If \(a \leq b\) and \(c \leq d\), then \(a + c \leq b + d\).

- **Proof**
  - Since \(a \leq b\), \(a + c \leq b + c\) (by OS); and since \(c \leq d\), \(b + c \leq b + d\) (by OS and A2)
  
  thus \(a + c \leq b + d\) (by OS).

2) If \(a \leq b\) and \(c \leq d\), then \(ac \leq bd\).

- **Proof**
  - Since \(a \leq b\) and \(c \leq d\), \(ac \leq bc\) (by OS).
  
  Since \(c \leq d\) and \(c \leq b\), \(bc \leq bd\) (by OS and M2),
  
  thus \(ac \leq bd\) (by OS).

2) If \(x > 0\), \(y > 0\), and \(x^2 < y^2\), then \(x < y\).

- **Proof** (by contradiction)
  - Suppose, instead that \(x > 0\), \(y > 0\), \(x^2 < y^2\), and \(x > y\),
  
  since \(0 \leq y \leq x\) and \(0 \leq x \leq x\), \(y^2 < x^2\) by Problem 2,
  
  and this gives a contradiction (using OS). Therefore if \(x > 0\), \(y > 0\), and \(x^2 < y^2\), then \(x < y\).

4) If \(a \leq x < y\), then \(a \leq x^n < y^n\) for all \(n \in \mathbb{N}\).

- **Proof** (by induction)
  - This is true for \(n = 1\), since \(a \leq x < y\).
  
  2) Assume that \(a \leq x^n < y^n\) for some \(n \in \mathbb{N}\).

  Since \(x > 0\), \(0 < x < x^n\) (by OS) so \(0 < x^{n+1}\) (by Th. 3.1, Part 4).

  Since \(x > 0\), \(x \cdot x^n < x \cdot y^n\) (by OS and M2) so \(x^{n+1} < x \cdot y^n\).

  Since \(y^n > 0\) and \(x < y\), \(x \cdot y^n < y^{n+1}\) (by OS),

  therefore \(x^{n+1} < y^{n+1}\) (by OS), so \(0 < x^{n+1} < y^{n+1}\).

  therefore \(a \leq x^n < y^n\) for all \(n \in \mathbb{N}\) by induction.
3. a) If \(0 < C < 1\), then \(C^n < C\) for any integer \(n > 1\).

   \[ \begin{align*}
   & \text{PF (by induction)} \\
   & 1) \text{ since } C > 0 \text{ and } C < 1, \quad C^2 < C \quad \text{(by 05 and 03)}, \quad \text{so } C^n < C \text{ for } n = 2, \\
   & 2) \text{ assume that } C^n < C \text{ for some } n \text{ in } \mathbb{N}, \text{ with } n > 2, \\
   & \quad \text{since } C > 0, \quad C^{n+1} < C^2 \quad \text{(by 05)}, \quad \text{so } C^{n+1} < C \quad \text{(by 03)} \quad \text{since } C^2 < C, \\
   & \text{therefore } C^n < C \text{ for any integer } n \geq 2. 
   \end{align*} \]

b) If \(C > 1\), then \(C^n > C\) for any integer \(n > 1\).

   \[ \begin{align*}
   & \text{PF (by induction)} \\
   & 1) \text{ since } C > 1 \text{ and } 1 > 0 \quad \text{(by 3.4A)}, \quad C > 0 \quad \text{(by 03)}; \\
   & \quad \text{so } C > 1 \quad \text{(by 05 and 03)}, \quad \text{so } C^n > C \text{ for } n = 2, \\
   & 2) \text{ assume that } C^n > C \text{ for some integer } n > 1, \\
   & \quad \text{since } C > 0, \quad C^{n+1} > C^2 \quad \text{(by 05)}, \quad \text{so } C^{n+1} > C \quad \text{(by 03)} \quad \text{since } C^2 > C, \\
   & \text{therefore } C^n > C \text{ for any integer } n \geq 2. 
   \end{align*} \]

6. If \(a^2 + b^2 = 0\), then \(a = 0\) and \(b = 0\).

   \[ \begin{align*}
   & \text{PF since } a^2 + b^2 = 0, \quad (a^2 + b^2) + (-b^2) = 0 + (-b^2) \quad \text{(using A4)} \quad \text{so} \\
   & \quad a^2 + (-b^2) = -b^2 \quad \text{(by A1 and A3)} \quad \text{and hence} \\
   & \quad a^2 = a^2 + 0 = -b^2 \quad \text{(by A4 and A3)}, \\
   & \quad \text{since } a^2 \geq 0 \text{ and } b^2 \geq 0 \quad \text{(by Th1, Part 4)} \quad \text{and} \\
   & \quad a^2 \geq 0 \quad \text{and} \quad b^2 \leq 0 \quad \text{(by Th2, Part 1)}, \\
   & \quad a^2 = 0 \quad \text{and} \quad b^2 = 0 \quad \text{(by Th1, Part 6)}. 
   \end{align*} \]

5a. PF: To show that \(a = 0\), assume instead that \(a^2 + b^2 = 0\) and \(a \neq 0\).

   \[ \begin{align*}
   & \text{then } a^2 \geq 0 \quad \text{(by Th2, Part 4)}, \quad \text{so } a^2 > 0 \quad \text{since otherwise} \\
   & \quad a^2 = 0 \quad \Rightarrow \quad a = 0 \quad \text{(by Th1, Part 6)}, \\
   & \text{therefore } a^2 + b^2 > 0 + b^2 = b^2 \quad \text{(by 04 and A3)}, \quad \text{and } b^2 > 0 \quad \text{(by Th2, Part 4)}, \\
   & \quad \text{so } a^2 + b^2 > 0 \quad \text{(by 03 and 02)}, \\
   & \text{this contradicts our assumption that } a^2 + b^2 = 0, \quad \text{so } a = 0. 
   \end{align*} \]

5i. Since \(a = 0\), \(b^2 = a + b^2 = a^2 + b^2 = 0 \quad \text{(by A3)} \quad \text{so } b = 0 \quad \text{(by Th1, Part 6)}. 

4. Cor 1 (to Th. 1)

   \[ \begin{align*}
   & \text{let } a \text{ be an element of an unbounded field } F. \\
   & \text{a)} \quad (-a) = a, \\
   & \text{b)} \quad (a^{-1})^{-1} = a \quad \text{if } a \neq 0. 
   \end{align*} \]

   \[ \begin{align*}
   & \text{PF } a) \quad (-a) + (-a) = 0 \quad \text{and} \quad a + (-a) = 0 \quad \text{(by A4)}, \quad \text{so } (-a) = a \quad \text{by Th. 1, Part 1}, \\
   & \text{b)} \quad (a^{-1})^{-1} (a^{-1}) = 1 \quad \text{and} \quad a (a^{-1}) = 1 \quad \text{(by M4)}, \quad \text{so } (a^{-1})^{-1} = a \quad \text{by Th. 1, Part 5}. 
   \end{align*} \]