Biorthogonal Wavelets of Maximum Coding Gain through Pseudoframes for Subspaces

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SPIE - August 15, 2006
Talk Overview

- Maximum coding gain
  - Goal to minimize the distortion arising from quantization
  - Design filters that are adapted to the statistics of the signal
Maximum coding gain

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- Definition: \( G_{SBC} \triangleq \frac{\sigma^2_{q,PCM}}{\sigma^2_{q,SBC}} \)
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- Pseudoframes for Subspaces (PFFS) biorthogonal wavelet filter construction
  - Parameterizes “out-of-subspace” components
  - Increased flexibility in filter design
  - Combines multiple design criteria into a single parameter

Coding gain of PFFS biorthogonal wavelets can be more than 2.5 times that of CDF spline wavelets
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Consider

- Signal $x(n)$ which is WSS with zero mean, variance $\sigma_x^2$
- Basic $b$-bit pulse-code-modulation (PCM) quantizer such as

$$x(n) \rightarrow Q \rightarrow x_q(n) = x(n) + q(n)$$

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- Signal $x(n)$ which is WSS with zero mean, variance $\sigma_x^2$
- Basic $b$-bit pulse-code-modulation (PCM) quantizer such as $x_q(n) = x(n) + q(n)$
- Number of bits is large enough to model the distortion from quantization $q(n)$ as additive white noise
- Well-known that

$$\sigma_{q,\text{PCM}}^2 = c \cdot 2^{-2b} \sigma_x^2$$

where $c$ is a constant of proportionality
SBC Quantization

Consider a two-channel subband coding (SBC) filter bank

- Input has PSD $S_{xx}(\gamma)$
- Filters are biorthogonal FIR
- In the absence of quantizers the filter bank has PR property
- Quantizers are ideal, scalar, uniform, midrise type
- $b_0$ bits for $Q_0$, $b_1$ bits for $Q_1$, and $b \triangleq \frac{1}{2}(b_0 + b_1)$
- Only source of error is due to noise from quantization
SBC Quantization Continued

- Under ideal conditions:  \[ \sigma_{q,SBC}^2 = c \cdot 2^{-2b} \phi^{1/2} \]
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where

\[ \Phi = \left( h^\mathcal{H} R_0 h \right) \left( \tilde{h}^\mathcal{H} R_1 \tilde{h}' \right) \left( \| \tilde{h} \|^2 \right) \left( \| h \|^2 \right) \]

\[ h = \{ h_n \}, \tilde{h} = \{ \tilde{h}_n \}, \tilde{h}' = \{ (-1)^n \tilde{h}_n \} \]

\( (\cdot)^\mathcal{H} \) denotes the Hermitian transpose, \( \| \cdot \| \) denotes the \( \ell^2 \)-norm, and ...
SBC Quantization Continued

- Under ideal conditions: \( \sigma^2_{q,SBC} = c \cdot 2^{-2b} \Phi^{1/2} \)

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h = \{ h_n \}, \quad \tilde{h} = \{ \tilde{h}_n \}, \quad \tilde{h}' = \{ (-1)^n \tilde{h}_n \}, \quad (\cdot)^H \text{ denotes the Hermitian transpose,} \quad \| \cdot \| \text{ denotes the } \ell^2\text{-norm, and ...}
\]

\[
R_k = \int_{-1/2}^{1/2} S_{xx}(\gamma) \begin{bmatrix}
1 & e^{-i2\pi\gamma} & \ldots & e^{-i2\pi(N_k-1)\gamma} \\
e^{i2\pi\gamma} & 1 & \ldots & e^{-i2\pi(N_k-2)\gamma} \\
& \vdots & \ddots & \vdots \\
e^{i2\pi(N_k-1)\gamma} & e^{i2\pi(N_k-2)\gamma} & \ldots & 1
\end{bmatrix} \, d\gamma
\]

- This matrix is positive definite, Hermitian and Toeplitz

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Definition of Coding Gain and Objective Function

- For fixed $b$, the subband coding gain compares the noise variances resulting from the PCM and SBC schemes:

$$G_{SBC} \triangleq \frac{\sigma^2_{q,PCM}}{\sigma^2_{q,SBC}} = \frac{\sigma^2_x}{\Phi^{1/2}}$$

- Goal is to maximize $G_{SBC}$

- For given signal $\sigma^2_x$ is fixed, thus $\Phi$ is our objective function to be minimized
PFFS is a notion of a frame-like expansion for a subspace $\chi$ of a separable Hilbert space $\mathcal{H}$.

$\{x_n\}$ is called a pseudoframe for the subspace $\chi$ with respect to $\{x_n^*\}$ if

$$f = \sum_n \langle f, x_n \rangle x_n^*$$

for any $f \in \chi$.

However, $\{x_n\}, \{x_n^*\} \not\subset \chi$. 
Constructing PFFS Dual Filters

- Start with a pair of biorthogonal scaling functions \( \varphi, \varphi^0 \in L^2(\mathbb{R}) \) where
  - \( \varphi \) generates an MRA \( \{V_j\} \) of \( L^2(\mathbb{R}) \)
  - \( \varphi^0 \in V_0 \) is the standard dual function of \( \varphi \)
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- Biorthogonal PFFS duals of \( \varphi \) are given by
  \[
  \tilde{\varphi} \triangleq \varphi^0 + \Delta \varphi
  \]
  where \( \Delta \varphi \in V_0^\perp \) is arbitrary
Now consider $\tilde{\varphi}$ sufficiently regular such that it obeys \[\tilde{\varphi} = \sum_n \tilde{h}_n \tilde{\varphi}_{1,n}\]

where $\tilde{\varphi}_{1,n} = \sqrt{2}\tilde{\varphi}(2t - n)$

Then, \[\tilde{h}_n = \langle \tilde{\varphi}, \varphi_{1,n} \rangle = \langle \varphi^0, \varphi_{1,n} \rangle + \langle \Delta \varphi, \varphi_{1,n} \rangle = h^0_n + \Delta h_n\]
Constructing PFFS Dual Filters

- Given biorthogonal filters $\mathbf{h} = \{ h_n \}, \mathbf{h}^0 = \{ h_0^0 \}$, the sequence $\Delta \mathbf{h} = \{ \Delta h_n \}$ derived from $V_0 \perp$ is uniquely determined.
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- Given biorthogonal filters \( h = \{ h_n \}, h^0 = \{ h^0_n \} \), the sequence \( \Delta h = \{ \Delta h_n \} \) derived from \( V_0^\perp \) is uniquely determined.

- With scale parameter \( \lambda \in \mathbb{R} \), the most simple PFFS dual pair is \( h = \{ h_n \}, \tilde{h} = \{ \tilde{h}_n \} \) where

\[
\tilde{h}_n = h^0_n + \lambda \Delta h_n
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Constructing PFFS Dual Filters

- Given biorthogonal filters $h = \{h_n\}, h^0 = \{h^0_n\}$, the sequence $\Delta h = \{\Delta h_n\}$ derived from $V_0^\perp$ is uniquely determined.

- With scale parameter $\lambda \in \mathbb{R}$, the most simple PFFS dual pair is $h = \{h_n\}, \tilde{h} = \{\tilde{h}_n\}$ where

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- More complicated parameter function can accommodate multiple design criteria.

- The regularity of $\tilde{h}_n$ is at least as good as that of $h^0_n$. 

Objective Function with Parameter $\lambda$

With the PFFS dual pair $h$ and $\tilde{h} = h^0 + \lambda \Delta h$ we have

$$
\Phi = \left( h^H R_0 h \right) \left( \tilde{h}'^H R_1 \tilde{h}' \right) \left( \| \tilde{h} \|^2 \right) \left( \| h \|^2 \right)
= K \left( A' \lambda^2 + B' \lambda + C' \right) \left( A \lambda^2 + B \lambda + C \right)
$$
With the PFFS dual pair \( h \) and \( \tilde{h} = h^0 + \lambda \Delta h \) we have

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\Phi = \left( h^\mathcal{H} R_0 h \right) \left( \tilde{h}^\mathcal{H} R_1 \tilde{h}' \right) \left( \| \tilde{h} \|^2 \right) \left( \| h \|^2 \right)
= K \left( A' \lambda^2 + B' \lambda + C' \right) \left( A \lambda^2 + B \lambda + C \right)
\]

where

\[
K = (h^\mathcal{H} R_0 h)(\|h\|^2),
A' = \Delta h'^\mathcal{H} R_1 \Delta h', \quad B' = 2 \text{Re} \left[ h^0'^\mathcal{H} R_1 \Delta h' \right], \quad C' = h^0'^\mathcal{H} R_1 h^0',
A = \| \Delta h \|^2, \quad B = 2 \text{Re} \left[ h^0^\mathcal{H} \Delta h \right], \quad C = \| h^0 \|^2,
\Delta h' = \{(-1)^n \Delta h_n\}, \quad h^0' = \{(-1)^n h^0_n\}
Details of Implementation

- Given biorthogonal pair \( h, h^0 \) and PSD \( S_{xx}(\gamma) \) we seek \( \lambda \) which minimizes \( \Phi \)
- Start with Cohen, Daubechies, and Feauveau (CDF) biorthogonal spline filters
- Input signal: auto-regressive type-2 (AR(2)) process with poles at \( 0.975e^{\pm i\theta} \)
- Pole at \( \theta \) degrees is indicative of where signal’s energy is focused in the frequency domain
Coding Gain of 4/20 filters for AR(2) Input vs. Pole $\theta$ [deg]
Ratio of $G_{\text{PFFS}}, G_{\text{CDF}}$ for AR(2) Input vs. Pole $\theta$ [deg]
Result of using $\lambda_{\text{min}} = -0.1465$ for AR(2), $\theta = 66.7^\circ$ Input

(a) Impulse Response vs Time, $n$

(b) Mag Response vs Normal Freq, $\gamma$

(c) Wavelet Function vs Time, $t$

(d) Scaling Function vs Time, $t$
Relationship with Bit Gain

For given level of distortion, bits gained using SBC over PCM:

\[ \Delta b = \frac{1}{2} \log_2 G_{SBC} \]
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Thus, the bits gained using PFFS over CDF:

$$\Delta b_{\text{PFFS}} - \Delta b_{\text{CDF}} = \frac{1}{2} \log_2 G_{\text{PFFS}} - \frac{1}{2} \log_2 G_{\text{CDF}}$$

$$= \frac{1}{2} \log_2 \frac{G_{\text{PFFS}}}{G_{\text{CDF}}}$$
Relationship with Bit Gain

- For given level of distortion, bits gained using SBC over PCM:
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- Thus, the bits gained using PFFS over CDF:
  \[ \Delta b_{PFFS} - \Delta b_{CDF} = \frac{1}{2} \log_2 G_{PFFS} - \frac{1}{2} \log_2 G_{CDF} = \frac{1}{2} \log_2 \frac{G_{PFFS}}{G_{CDF}} \]

- For \( \theta = 66.7^\circ \): \( G_{PFFS} = 3.82 \) and \( G_{CDF} = 1.51 \)

- Thus, PFFS gains 0.6691 bits per sample over CDF for AR(2) signals with \( \theta = 66.7^\circ \)
Result of using $\lambda_{\text{min}} = -0.0360$ for White Noise Input

(a) Impulse Response vs Time, $n$

(b) Mag Response vs Normal Freq, $\gamma$

(c) Wavelet Function vs Time, $t$

(d) Scaling Function vs Time, $t$
Conclusions/Future Work

- PFFS model can easily incorporate the design criterion of maximizing coding gain
- Constrained problem became unconstrained/more flexible
- For AR(2) signals, the PFFS biorthogonal wavelet filters had a higher coding gain than the same length filters of CDF
- For AR(2) signals $\theta = 66.7^\circ$ the coding gain and resulting bit gain of PFFS were significantly better than that of CDF
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- For AR(2) signals $\theta = 66.7^\circ$ the coding gain and resulting bit gain of PFFS were significantly better than that of CDF
- Potential to incorporate several design criteria
- PFFS model needs to be tested with other classes of inputs
- Need to see how different length filters affect the coding gain
The end.
PSD for AR(2), $\theta = 66.7^\circ$ and White Noise Inputs
Objective Function $\Phi$ vs. $\lambda$ for AR(2), $\theta = 66.7^\circ$ Input
Objective Function $\Phi$ vs. $\lambda$ for White Noise Input