57. \( \frac{dN}{dt} = \frac{2(1+ \sin(2\pi t))}{f(t)} N(t) \)

\[
\int \frac{dN}{N} = \int 2(1+ \sin(2\pi t)) \, dt
\]

\( \ln |N| = 2t - \frac{2}{2\pi} \cos(2\pi t) + C \)

\[ 
\ln |N| = 2t - \frac{1}{\pi} \cos(2\pi t) + C
\]

\[ 
N = Ce^{2t - \frac{1}{\pi} \cos(2\pi t)} \quad N(t) = 5e^{2t - \frac{1}{\pi} \cos(2\pi t)} +
\]

\[ 
N(0) = 5 = Ce^{-1} \quad C = 5e
\]

19. \( \frac{1}{N} \frac{dN}{dt} = r \)

\( N(t) = Ce^{rt} \quad y = mx + b \)

(a) \[ \int \frac{dN}{N} = \int r \, dt \]

\[ \ln |N| = rt + C \quad x = t \]

\[ N(t) = Ce^{rt} \quad y = mx + C \]

(c) \( r \) is the slope
1) Set \( y = \ln |N| \)
\[ x = t \]

2) Find a linear regression for \( y \) vs. \( x \).

3) Slope is \( r \)

\[ y = mx + b \]
Linear first-order differential equations.

\[ \frac{dx}{dt} + p(t)x = q(t) \]

\[ \text{functions of independent variable} \]

\[ I(t) = \int p(t) \, dt \]

Multiply by integrating factor

\[ e^{\int p(t) \, dt} \left( \frac{dx}{dt} + p(t)x \right) = \int q(t) \, dt \]

\[ e^{\int p(t) \, dt} \frac{dx}{dt} + e^{\int p(t) \, dt} p(t)x(t) = e^{\int p(t) \, dt} q(t) \]

\[ \frac{d}{dt} e^{\int p(t) \, dt} x(t) = e^{\int p(t) \, dt} q(t) \]

\[ \int \frac{d}{dt} \left( e^{\int p(t) \, dt} x(t) \right) \, dt = \int e^{\int p(t) \, dt} q(t) \, dt \]

\[ x(t) = e^{-\int p(t) \, dt} \left( C + \int e^{\int p(t) \, dt} q(t) \, dt \right) \]
Solve for $y(x)$

\[
\frac{dy}{dx} + 2y = 5 \implies p(t) = 2 \\
q(t) = 5
\]

Find the integrating factor

\[
I(t) = e^{\int 2 \, dx} = e^{2x}
\]

\[
e^{2x} \frac{dy}{dx} + e^{2x} 2y = 5e^{2x}
\]

\[
\int \frac{d}{dx} (e^{2x} y) \, dx = \int 5e^{2x} \, dx
\]

\[
e^{2x} y = \frac{5}{2} e^{2x} + C
\]

\[
y = \frac{5}{2} e^{2x} + Ce^{-2x}
\]

Evaluate:

\[
y' + 3x^2 y = x^2 \implies p(x) = 3x^2 \\
q(x) = x^2
\]

\[
e^{\int 3x^2 \, dx} = e^{x^3}
\]

\[
e^{x^3} y' + 3x^2 e^{x^3} y = e^{x^3} x^2
\]

\[
\left( e^{x^3} y \right)' = e^{x^3} x^2
\]

\[
\int x^2 e^{x^3} \, dx
\]

\[
u = x^3
\]

\[
du = 3x^2
\]

\[
e^{x^3} y = \frac{1}{3} e^{x^3} + C
\]
Ex. \((1 + x^2)y' + xy + x^3 + x = 0\)

\[
\frac{1}{1 + x^2} \left[ (1 + x^2)y' + xy = -x - x^3 \right]
\]

\[y' + \frac{x}{1 + x^2} y = -\frac{x - x^3}{1 + x^2}, \quad p(x) = \frac{x}{1 + x^2}, \quad q(x) = -\frac{x - x^3}{1 + x^2}\]

\[
\int \frac{-x}{1 + x^2} \, dx = \frac{1}{2} \ln |1 + x^2|, \quad I(x) = e^{\frac{1}{2} \ln |1 + x^2|} = e^{\ln (1 + x^2)^{1/2}} = (1 + x^2)^{1/2}
\]

\[
\int \sqrt{1 + x^2} \, y' + \sqrt{1 + x^2} \frac{x}{1 + x^2} \cdot y = \sqrt{1 + x^2} \frac{-x - x^3}{1 + x^2}
\]

\[
\left(\sqrt{1 + x^2} \, y\right)' = \frac{-x - x^3}{\sqrt{1 + x^2}}
\]

\[
\sqrt{1 + x^2} \, y = \int \frac{-x - x^3}{\sqrt{1 + x^2}} \, dx
\]

\[
\int -\frac{x}{\sqrt{1 + x^2}} \, dx = \int \frac{x^2}{\sqrt{1 + x^2}} \, dx - \int \frac{1}{\sqrt{u}} \, du - \frac{1}{2} x^2
\]

\[
-\frac{1}{2} 2 \sqrt{1 + x^2} \quad u = 1 + x^2 \quad \int \frac{1}{\sqrt{u}} (u - 1) \, du
\]

\[
du = 2 \sqrt{u} \, dx \quad \int \frac{1}{2} \left( \sqrt{u} - \frac{1}{\sqrt{u}} \right) \, du
\]

\[
x^2 = u - 1 \quad \frac{1}{2} \cdot \frac{2}{3} u^{3/2} - \frac{1}{2} \cdot 2 \sqrt{u}
\]

\[
-\frac{1}{3} (1 + x^2)^{3/2} \quad \frac{1}{3} (1 + x^2)^{3/2} - (1 + x^2)^{1/2}
\]

\[
(1 + x^2)^{1/2} y = -\frac{1}{3} (1 + x^2)^{3/2} + C
\]

\[
y = -\frac{1}{3} (1 + x^2)^{3/2} + C (1 + x^2)^{-1/2}
\]
Ex: \( \cos^2 x \sin x \frac{dy}{dx} + y \cos^3 x = 1 \)

\[
\frac{dy}{dx} + \frac{\cos x}{\sin x} y = \frac{1}{\cos^2 x \sin x} \\
\cot x
\]

\[
\int \frac{\cos x}{\sin x} \, dx = \int \frac{1}{u} \, du = \ln |u|
\]

\( u = \sin x \)

\( du = \cos x \)

\[
\ln |\sin x| = \sin x
\]

\[
\frac{dy}{dx} \sin x + \sin x \cos x \frac{dy}{dx} \frac{\cos x}{\sin x} y = \frac{1}{\cos^2 x} \\
(\sin x \cdot y)' = \sec^2 x \\
\frac{\sin x}{\cos x} \frac{1}{\sin} \\
\sin x \cdot y = \tan x + C \\
y = \sec x + C \csc x
\]
8.2 Equilibria and Stability

\[
\frac{dy}{dx} = g(y)
\]

We want to analyze asymptotic behavior without solving the equation.

\[
\frac{dN}{dt} = rN \left( 1 - \frac{N}{K} \right) = g(N)
\]

\[
N(t) = \frac{K}{1 + \left( \frac{K}{N_0} - 1 \right) e^{-rt}}
\]

\[
\lim_{t \to \infty} N(t) = K \leftarrow \text{equilibrium point}
\]

if \( N_0 \neq 0 \)

if \( N_0 = 0 \) \( \lim_{t \to \infty} N(t) = 0 \leftarrow \text{equilibrium point} \)

\[
g(N) = rN \left( 1 - \frac{N}{K} \right)
\]

\[
g(0) = r \cdot 0 \left( 1 - \frac{0}{K} \right) = 0
\]

\[
g(K) = r \cdot K \left( 1 - \frac{K}{K} \right) = 0
\]

If \( g(\hat{N}) = 0 \), then \( \hat{N} \) is an equilibrium point of

\[
\frac{dN}{dt} = g(N)
\]

\[\uparrow \quad \text{unstable} \quad \rightarrow \quad \text{stable} \]
\[
\frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right)
\]

\[
rN - \frac{r}{K}N^2 = 0
\]

For \(N > K\), \(\frac{dN}{dt} = g(N) < 0\)

change in population is negative \(\Rightarrow\) population is decreasing

For \(N < K\), \(\frac{dN}{dt} = g(N) > 0\)

change in population is positive \(\Rightarrow\) population is increasing

If \(\hat{N}\) is an equilibrium point, then
\(\hat{N}\) is stable if \(g'(\hat{N}) < 0\)
\(\hat{N}\) is unstable if \(g'(\hat{N}) > 0\)

\[N(0) = \hat{N}\]
The Levins model models insect colony populations

\[
\frac{dp}{dt} = cp(1-p) - mp \\
p \text{ is the population }
\]

\[
c \text{ colonization rate}
\]

\[
m \text{ mortality}
\]

\[
g(p) = cp(1-p) - mp
\]

\[
= cp(1 - \frac{m}{c} - p)
\]

To find equilibrium points set

\[
g(p) = 0
\]

\[
cp(1 - \frac{m}{c} - p) = 0
\]

\[
cp = 0 \quad 1 - \frac{m}{c} - \hat{p} = 0
\]

\[
\hat{p} = 0 \quad \hat{p} = 1 - \frac{m}{c}
\]

\[
\hat{p}_1 = 1 - \frac{m}{c}
\]

\[
m > c \Rightarrow \hat{p}_1 < 0
\]

\[
m < c \Rightarrow \hat{p}_2 > 0
\]
\[ g(p) = cp(1-p) - mp \]

\[ m > c \]
\[ g'(p) = c - 2cp - m \]
\[ \hat{p}_0 = 0 \]
\[ g'(-c) = c - m < 0 \Rightarrow \hat{p}_0 \text{ is stable} \]
\[ \hat{p}_e = 1 - \frac{m}{c} < 0 \text{ no meaning} \]

\[ m < c \]
\[ g'(\hat{p}_e) = c - m > 0 \Rightarrow \hat{p}_e \text{ is unstable} \]
\[ g'(\hat{p}_e) = c - 2c(1 - \frac{m}{c}) - m \]
\[ = c - 2c + 2m - m \]
\[ = m - c < 0 \Rightarrow \hat{p}_e \text{ is stable} \]

The Allee Effect
\[ \frac{dN}{dt} = rN(N-a)(1-\frac{N}{K}) \quad r, K, a > 0 \quad 0 < a < K \]

equilibrium points
\[ g(N) = rN(N-a)(1-\frac{N}{K}) \]
\[ g(\hat{N}) = r\hat{N}(\hat{N}-a)(1-\frac{\hat{N}}{K}) = 0 \]
\[ r\hat{N} = 0 \quad \hat{N}-a=0 \quad 1-\frac{\hat{N}}{K} = 0 \]
\[ \hat{N} = 0 \quad \hat{N} = a \quad \hat{N} = K \]
\[ g'(0) = \frac{r}{K}(-ak)<0 \]
\[ g'(a) = \frac{r}{K}a(K-a) = 0 \]
\[ g'(k) = \frac{r}{K}k(k-a) < 0 \]
\[ r \hat{N} = 0 \quad \hat{N}-a=0 \quad 1-\frac{\hat{N}}{K} = 0 \]
\[ \hat{N}=0 \quad \hat{N}=a \quad \hat{N}=K \]
\[ K \text{ is stable} \]
\[ g'(N) = r \left( 2N + \frac{2a}{K}N - \frac{3N^2}{K} - a \right) \]
\[ = \frac{r}{K} \left( 2NK + 2aN - 3N^2 - aK \right) \]
Solution curves

Phase portrait
• - stable point
○ - unstable point
○/○ - semistable

\[ g(N) = 0 \implies \text{unstable?} \]

- semistable