Key Terms

Reduced row echelon form
Leading one
Row echelon form
Elementary row operation
Row equivalent
Reduced row echelon form of a matrix
Row echelon form of a matrix
Gauss–Jordan reduction
Gaussian elimination
Back substitution
Consistent linear system
Inconsistent linear system
Homogeneous system
Trivial solution
Nontrivial solution
Bit linear systems

1.6 Exercises

In Exercises 1 through 8, determine whether the given matrix is in reduced row echelon form, row echelon form, or neither.

1. \[
\begin{bmatrix}
1 & 0 & 0 & 0 & -3 \\
0 & 0 & 1 & 0 & 4 \\
0 & 0 & 0 & 1 & 2 \\
\end{bmatrix}
\]

2. \[
\begin{bmatrix}
0 & 1 & 0 & 0 & 5 \\
0 & 0 & 1 & 0 & -4 \\
0 & 0 & 0 & 1 & 3 \\
\end{bmatrix}
\]

3. \[
\begin{bmatrix}
1 & 0 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

4. \[
\begin{bmatrix}
0 & 1 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

5. \[
\begin{bmatrix}
1 & 2 & 3 & 1 \\
0 & 1 & 2 & 3 \\
0 & 0 & 1 & -4 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

6. \[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

7. \[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & -3 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

8. \[
\begin{bmatrix}
0 & 1 & 0 & 0 & 5 \\
0 & 0 & 1 & 0 & 4 \\
0 & 1 & 0 & -2 & 3 \\
\end{bmatrix}
\]

9. Let \( A = \begin{bmatrix} 1 & 0 & 3 \\ -3 & 1 & 4 \\ 4 & 2 & 2 \\ 5 & -1 & 5 \end{bmatrix} \)

Find the matrices obtained by performing the following elementary row operations on \( A \).

(a) Interchanging the second and fourth rows

(b) Multiplying the third row by 3

(c) Adding \((-3)\) times the first row to the fourth row

10. Let \( A = \begin{bmatrix} 2 & 0 & 4 & 2 \\ 3 & -2 & 5 & 6 \\ -1 & 3 & 1 & 1 \end{bmatrix} \)

Find the matrices obtained by performing the following elementary row operations on \( A \).

(a) Interchanging the second and third rows

(b) Multiplying the second row by \((-4)\)

(c) Adding \(2\) times the third row to the first row

11. Find three matrices that are row equivalent to \( A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 5 & 2 & -3 & 4 \end{bmatrix} \)

12. Find three matrices that are row equivalent to \( A = \begin{bmatrix} 4 & 3 & 7 & 5 \\ -1 & 2 & -1 & 3 \\ 2 & 0 & 1 & 4 \end{bmatrix} \)

In Exercises 13 through 16, find a row echelon form of the given matrix.

13. \[
\begin{bmatrix}
0 & -1 & 2 & 3 \\
2 & 3 & 4 & 5 \\
1 & 3 & -1 & 2 \\
3 & 2 & 4 & 1 \\
\end{bmatrix}
\]

14. \[
\begin{bmatrix}
1 & -2 & 0 & 2 \\
2 & -3 & -1 & 5 \\
1 & 3 & 2 & 5 \\
1 & 1 & 0 & 2 \\
2 & -6 & -2 & 1 \\
\end{bmatrix}
\]

15. \[
\begin{bmatrix}
1 & 2 & -3 & 1 \\
-1 & 0 & 3 & 4 \\
0 & 1 & 2 & -1 \\
2 & 3 & 0 & -3 \\
\end{bmatrix}
\]

16. \[
\begin{bmatrix}
2 & -1 & 0 & 1 & 4 \\
1 & -2 & 1 & 4 & -3 \\
5 & -4 & 1 & 6 & 5 \\
-7 & 8 & -3 & -14 & 1 \\
\end{bmatrix}
\]
17. For each of the matrices in Exercises 13 through 16, find the reduced row echelon form of the given matrix.

18. Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 2 & 1 & -2 \end{bmatrix}.$$  

In each part, determine whether \( \mathbf{x} \) is a solution to the linear system \( A\mathbf{x} = \mathbf{b} \).

(a) \( \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \)  
(b) \( \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \)

(c) \( \mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix} \)

(d) \( \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \)

19. Let

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 1 & 3 & 0 & 2 \\ -1 & 2 & 1 & 3 \end{bmatrix}.$$  

In each part, determine whether \( \mathbf{x} \) is a solution to the homogeneous system \( A\mathbf{x} = \mathbf{0} \).

(a) \( \mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ -1 \\ 2 \\ 2 \end{bmatrix} \)

(b) \( \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \)

(c) \( \mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix} \)

(d) \( \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \)

In Exercises 20 through 22, find all solutions to the given linear system.

20. (a) \( x + y + 2z = -1 \)
\( x - 2y + z = -5 \)
\( 3x + y + z = 3 \)

(b) \( x + y + 3z + 2w = 7 \)
\( 2x - y + 4w = 8 \)
\( 3y + 6z = 8 \)

(c) \( x + 2y - 4z = 3 \)
\( x - 2y + 3z = -1 \)
\( 2x + 3y - z = 5 \)
\( 4x + 3y - 2z = 7 \)
\( 5x + 2y - 6z = 7 \)

(d) \( x + y + z = 0 \)
\( x + z = 0 \)
\( 2x + y - 2z = 0 \)
\( x + 5y + 5z = 0 \)

21. (a) \( x + y + 2z + 3w = 13 \)
\( x - 2y + z + w = 8 \)
\( 3x + y + z - w = 1 \)

(b) \( x + y + z = 1 \)
\( x + y - 2z = 3 \)
\( 2x + y + z = 2 \)

(c) \( 2x + y + z - 2w = 1 \)
\( 3x - 2y + z - 6w = 2 \)
\( x + y - z - w = 1 \)
\( 6x + 2z - 9w = 3 \)
\( 5x - y + 2z - 8w = 3 \)

(d) \( x + 2y + 3z - w = 0 \)
\( 2x + y - z + w = 3 \)
\( x - y + w = -2 \)

22. (a) \( 2x - y + z = 3 \)
\( x - 3y + z = 4 \)
\( -5x - 2z = -5 \)

(b) \( x + y + z + w = 6 \)
\( 2x + y - z = 3 \)
\( 3x + y + 2w = 6 \)

(c) \( 2x - y + z = 3 \)
\( 3x + y - 2z = 2 \)
\( x + 5y + 7z = 13 \)
\( x - 7y - 5z = 12 \)

(d) \( x + 2y - z = 0 \)
\( 2x + y + z = 0 \)
\( 5x + 7y + z = 0 \)

In Exercises 23 through 26, find all values of \( a \) for which the resulting linear system has (a) no solution, (b) a unique solution, and (c) infinitely many solutions.

23. \( x + y - z = 2 \)
\( x + 2y + z = 3 \)
\( x + y + (a^2 + 5)z = a \)

24. \( x + y + z = 2 \)
\( 2x + 3y + z = 5 \)
\( 2x + 3y + (a^2 - 1)z = a + 1 \)

25. \( x + y + z = 2 \)
\( x + 2y + z = 3 \)
\( x + y + (a^2 - 5)z = a \)

26. \( x + y + z = 3 \)
\( x + (a^2 - 8)y + a \)

In Exercises 27 through 30, solve the linear system with the given augmented matrix.

27. (a) \( \begin{bmatrix} 1 & 1 & 1 & : & 0 \\ 1 & 1 & 0 & : & 3 \\ 0 & 1 & 1 & : & 1 \end{bmatrix} \)
always compute values of \( x, y, \) and \( z \) for which
\[
f \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.
\]

34. Let \( f : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be the matrix transformation defined by
\[
f \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.
\]
Find an equation relating \( a, b, \) and \( c \) so that we can always compute values of \( x, y, \) and \( z \) for which
\[
f \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.
\]

In Exercises 35 and 36, solve the linear systems \( Ax = b_1 \) and \( Ax = b_2 \) separately and then by obtaining the reduced row echelon form of the augmented matrix \([A: b_1 b_2]\). Compare your answers.

35. \( A = \begin{bmatrix} 1 & -1 & 3 \\ -2 & -3 & 5 \end{bmatrix}, b_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, b_2 = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \)

36. \( A = \begin{bmatrix} 1 & -2 & 0 \\ -3 & 2 & 3 \\ 4 & -2 & 3 \end{bmatrix}, b_1 = \begin{bmatrix} 3 \\ -7 \end{bmatrix}, b_2 = \begin{bmatrix} -4 \\ -10 \end{bmatrix} \)

In Exercises 37 and 38, let
\[
A = \begin{bmatrix} 1 & 0 & 5 \\ 1 & 1 & 1 \\ 0 & 1 & -4 \end{bmatrix}
\]

37. Find a nontrivial solution to the homogeneous system \((-4I_3 - A)x = 0\).

38. Find a nontrivial solution to the homogeneous system \((2I_3 - A)x = 0\).

39. Find an equation relating \( a, b, \) and \( c \) so that the linear system
\[
\begin{align*}
x + 2y - 3z &= a \\
2x + 3y + 3z &= b \\
5x + 9y - 6z &= c
\end{align*}
\]
is consistent for any values of \( a, b, \) and \( c \) that satisfy that equation.

40. Find an equation relating \( a, b, \) and \( c \) so that the linear system
\[
\begin{align*}
2x + 2y + 3z &= a \\
3x - y + 5z &= b \\
x - 3y + 2z &= c
\end{align*}
\]
is consistent for any values of \( a, b, \) and \( c \) that satisfy that equation.

*This type of problem will play a key role in Chapter 8.*
41. Find a $2 \times 1$ matrix $x$ with entries not all zero such that $Ax = 4x$, where

$$A = \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix}.$$  

[Hint: Rewrite the matrix equation $Ax = 4x$ as $4x - Ax = (4I - A)x = 0$ and solve the homogeneous system.]

42. Find a $2 \times 1$ matrix $x$ with entries not all zero such that $Ax = 3x$, where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^*.$$  

43. Find a $3 \times 1$ matrix with entries not all zero such that $Ax = 3x$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}.$$  

44. Find a $3 \times 1$ matrix $x$ with entries not all zero such that $Ax = 1x$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}^*.$$  

In Exercises 45 and 46, solve the given linear system and write the solution $x$ as $x = x_p + x_h$, where $x_p$ is a particular solution to the given system and $x_h$ is a solution to the associated homogeneous system.

45.  
\begin{align*}
x + 2y - z - 2w &= 2 \\
2x + y - 2z + 3w &= 2 \\
x + 2y + 3z + 4w &= 5 \\
4x + 5y - 4z - w &= 6
\end{align*}

46.  
\begin{align*}
x - y - 2z + 3w &= 4 \\
3x + 2y - z + 2w &= 5 \\
- y - 7z + 9w &= -2
\end{align*}

In Exercises 47 and 48, find the quadratic polynomial that interpolates the given points.

47.  
\begin{align*}
(1, 2), (3, 3), (5, 8)
\end{align*}

48.  
\begin{align*}
(1, 5), (2, 12), (3, 44)
\end{align*}

In Exercises 49 and 50, find the cubic polynomial that interpolates the given points.

49.  
\begin{align*}
(-1, -6), (1, 0), (2, 8), (3, 34)
\end{align*}

50.  
\begin{align*}
(-2, 2), (-1, 2), (1, 2), (2, 10)
\end{align*}

51. A furniture manufacturer makes chairs, coffee tables, and dining-room tables. Each chair requires 10 minutes of sanding, 6 minutes of staining, and 12 minutes of varnishing. Each coffee table requires 12 minutes of sanding, 8 minutes of staining, and 12 minutes of varnishing. Each dining-room table requires 15 minutes of sanding, 12 minutes of staining, and 18 minutes of varnishing. The sanding bench is available 16 hours per week, the staining bench 11 hours per week, and the varnishing bench 18 hours per week. How many (per week) of each type of furniture should be made so that the benches are fully utilized?

52. A book publisher publishes a potential best seller in three different bindings: paperback, book club, and deluxe. Each paperback book requires 1 minute for sewing and 2 minutes for gluing. Each book club book requires 2 minutes for sewing and 4 minutes for gluing. Each deluxe book requires 3 minutes for sewing and 5 minutes for gluing. If the sewing plant is available 6 hours per day and the gluing plant is available 11 hours per day, how many books of each type can be produced per day so that the plants are fully utilized?

53. (Calculus Required) Construct a linear system of equations to determine a quadratic polynomial $p(x) = ax^2 + bx + c$ that satisfies the conditions $p(0) = f(0), p'(0) = f'(0)$, and $p''(0) = f''(0)$, where $f(x) = e^{2x}$.

54. (Calculus Required) Construct a linear system of equations to determine a quadratic polynomial $p(x) = ax^2 + bx + c$ that satisfies the conditions $p(1) = f(1), p'(1) = f'(1)$, and $p''(1) = f''(1)$, where $f(x) = xe^{x-1}$.

55. Determine the temperatures at the interior points $T_i$, $i = 1, 2, 3, 4$ for the plate shown in the figure. (See Example 17.)

In Exercises 56 through 59, solve the bit linear systems.

56. (a) $x + y + z = 0$  
\begin{align*}
y + z &= 1 \\
x + y &= 1
\end{align*}

(b) $x + y + z = 1$  
\begin{align*}
x + y &= 0 \\
y + z &= 1
\end{align*}

57. (a) $x + y + w = 0$  
\begin{align*}
x + z + w &= 1 \\
y + z + w &= 1 \\
x + y + z + w &= 0
\end{align*}

(b) $x + y + z = 0$  
\begin{align*}
x + y + z &= 1 \\
x + y + z + w &= 0
\end{align*}

58. (a) $x + y + z = 1$  
\begin{align*}
x + y + z &= 0 \\
x + y + z + w &= 0
\end{align*}

(b) $x + y + z = 0$  
\begin{align*}
x + y + z &= 1 \\
x + y + z + w &= 0
\end{align*}

*This type of problem will play a key role in Chapter 8.*
59. Solve the bit linear system $Ax = c$, where
   \[
   (a) \quad A = \begin{bmatrix}
   1 & 1 & 0 \\
   0 & 1 & 0 \\
   1 & 1 & 1
   \end{bmatrix},
   c = \begin{bmatrix}
   0 \\
   1 \\
   0
   \end{bmatrix}
   \]
   \[
   (b) \quad A = \begin{bmatrix}
   1 & 1 & 0 & 1 \\
   0 & 1 & 1 & 1 \\
   0 & 0 & 1 & 1 \\
   0 & 1 & 1 & 0
   \end{bmatrix},
   c = \begin{bmatrix}
   1 \\
   0 \\
   0 \\
   0
   \end{bmatrix}
   \]

Theoretical Exercises

T.1. Show that properties (a), (b), and (c) alone [excluding (d)] of the definition of the reduced row echelon form of a matrix $A$ imply that if a column of $A$ contains a leading entry of some row, then all other entries in that column below the leading entry are zero.

T.2. Show that
   (a) Every matrix is row equivalent to itself.
   (b) If $A$ is row equivalent to $B$, then $B$ is row equivalent to $A$.
   (c) If $A$ is row equivalent to $B$ and $B$ is row equivalent to $C$, then $A$ is row equivalent to $C$.


T.4. Show that the linear system $Ax = b$, where $A$ is $n \times n$, has no solution if and only if the reduced row echelon form of the augmented matrix has a row whose first $n$ elements are zero and whose $(n+1)^{st}$ element is 1.

T.5. Let
   \[
   A = \begin{bmatrix}
   a & b \\
   c & d
   \end{bmatrix}.
   \]
   Show that $A$ is row equivalent to $I_2$ if and only if $ad - bc \neq 0$.

T.6. (a) Let
   \[
   A = \begin{bmatrix}
   a & b \\
   ka & kb
   \end{bmatrix}.
   \]
   Use Exercise T.5 to determine if $A$ is row equivalent to $I_2$.
   (b) Let $A$ be a $2 \times 2$ matrix with a row consisting entirely of zeros. Use Exercise T.5 to determine if $A$ is row equivalent to $I_2$.

T.7. Determine the reduced row echelon form of the matrix
   \[
   \begin{bmatrix}
   \cos \theta & \sin \theta \\
   -\sin \theta & \cos \theta
   \end{bmatrix}
   \]

T.8. Let
   \[
   A = \begin{bmatrix}
   a & b \\
   c & d
   \end{bmatrix}.
   \]

Show that the homogeneous system $Ax = 0$ has only the trivial solution if and only if $ad - bc \neq 0$.

T.9. Let $A$ be an $n \times n$ matrix in reduced row echelon form. Show that if $A$ is not equal to $I_n$, then $A$ has a row consisting entirely of zeros.

T.10. Show that the values of $\lambda$ for which the homogeneous system
   \[
   (a - \lambda)x + by = 0
   \]
   \[
   cx + (d - \lambda)y = 0
   \]
   has a nontrivial solution satisfy the equation
   \[
   (a - \lambda)(d - \lambda) - bc = 0. \quad \text{(Hint: See Exercise T.8.)}
   \]

T.11. Let $u$ and $v$ be solutions to the homogeneous linear system $Ax = 0$.
   (a) Show that $u + v$ is a solution.
   (b) Show that $u - v$ is a solution.
   (c) For any scalar $r$, show that $ru$ is a solution.
   (d) For any scalars $r$ and $s$, show that $ru + sv$ is a solution.

T.12. Show that if $u$ and $v$ are solutions to the linear system $Ax = b$, then $u - v$ is a solution to the associated homogeneous system $Ax = 0$.

T.13. Let $Ax = b$, $b \neq 0$, be a consistent linear system.
   (a) Show that if $x_p$ is a particular solution to the given nonhomogeneous system and $x_h$ is a solution to the associated homogeneous system $Ax = 0$, then $x_p + x_h$ is a solution to the given system $Ax = b$.
   (b) Show that every solution $x$ to the nonhomogeneous linear system $Ax = b$ can be written as $x_p + x_h$, where $x_p$ is a particular solution to the given nonhomogeneous system and $x_h$ is a solution to the associated homogeneous system $Ax = 0$. \textit{[Hint:} Let $x = x_p + (x - x_p).$\textit{]}

T.14. Justify the second remark following Example 12.
Key Terms
- Inverse
- Nonsingular (or invertible) matrix
- Singular (or noninvertible) matrix

1.7 Exercises

In Exercises 1 through 4, use the method of Examples 2 and 3.

1. Show that \( \begin{bmatrix} 2 & 1 \\ -2 & 3 \end{bmatrix} \) is nonsingular.

2. Show that \( \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \) is singular.

3. Is the matrix
\[ \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \]
singular or nonsingular? If it is nonsingular, find its inverse.

4. Is the matrix
\[ \begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & 3 \\ 2 & 2 & 1 \end{bmatrix} \]
singular or nonsingular? If it is nonsingular, find its inverse.

In Exercises 5 through 10, find the inverses of the given matrices, if possible.

5. (a) \[ \begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix} \]  \hspace{1cm} (b) \[ \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \]
   (c) \[ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \]

6. (a) \[ \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \]  \hspace{1cm} (b) \[ \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix} \]
   (c) \[ \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & 0 \\ 0 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix} \]

7. (a) \[ \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \]  \hspace{1cm} (b) \[ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 1 \\ 5 & 9 & 6 \end{bmatrix} \]
   (c) \[ \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \]

8. (a) \[ \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \]  \hspace{1cm} (b) \[ \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix} \]

9. (a) \[ \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 3 & 1 \end{bmatrix} \]  \hspace{1cm} (b) \[ \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} \]
   (c) \[ \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix} \]

10. (a) \[ \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \]  \hspace{1cm} (b) \[ \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \]
    (c) \[ \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \]

11. Which of the following linear systems have a nontrivial solution?
   (a) \[ \begin{align*}
   x + 2y + 3z &= 0 \\
   2y + 2z &= 0 \\
   x - 2y - 3z &= 0 \\
   x + 2y + 3z &= 0 \\
   -3x - y + 2z &= 0
   \end{align*} \]

12. Which of the following linear systems have a nontrivial solution?
   (a) \[ \begin{align*}
   x + y + 2z &= 0 \\
   2x + y + z &= 0 \\
   3x - y + z &= 0 \\
   2x - 2y + 2z &= 0
   \end{align*} \]
   (b) \[ \begin{align*}
   x - y + z &= 0 \\
   2x + y &= 0 \\
   2x - 2y + 2z &= 0
   \end{align*} \]
   (c) \[ \begin{align*}
   2x - y + 5z &= 0 \\
   3x + 2y - 3z &= 0 \\
   x - y + 4z &= 0
   \end{align*} \]

13. If \( A^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \), find \( A \).

14. If \( A^{-1} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} \), find \( A \).

15. Show that a matrix that has a row or column consisting entirely of zeros must be singular.

16. Find all values of \( a \) for which the inverse of
\[ A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & a \end{bmatrix} \]
exists. What is \( A^{-1} \)?
17. Consider an industrial process whose matrix is

\[ A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix}. \]

Find the input matrix for each of the following output matrices:
(a) \[ \begin{bmatrix} 30 \\ 20 \\ 10 \end{bmatrix} \quad \text{(b) } \begin{bmatrix} 12 \\ 8 \\ 14 \end{bmatrix} \]

18. Suppose that \( A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \).

(a) Find \( A^{-1} \).

(b) Find \( (A^T)^{-1} \). How do \( (A^T)^{-1} \) and \( A^{-1} \) compare?

19. Is the inverse of a nonsingular symmetric matrix always symmetric? Explain.

20. (a) Is \( (A + B)^{-1} = A^{-1} + B^{-1} \) for all \( A \) and \( B \)?

(b) Is \( (cA)^{-1} = \frac{1}{c} A^{-1} \), for \( c \neq 0 \)?

21. For what values of \( \lambda \) does the homogeneous system

\[
(\lambda - 1)x + 2y = 0 \\
2x + (\lambda - 1)y = 0
\]

have a nontrivial solution?

22. If \( A \) and \( B \) are nonsingular, are \( A + B, A - B, \) and \( -A \) nonsingular? Explain.

23. If \( D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \), find \( D^{-1} \).

24. If \( A^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} \) and \( B^{-1} = \begin{bmatrix} 2 & 5 \\ 3 & -2 \end{bmatrix} \), find \( (AB)^{-1} \).

25. Solve \( Ax = b \) for \( x \) if

\[
A^{-1} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.
\]

26. Let \( A \) be a \( 3 \times 3 \) matrix. Suppose that \( x = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \) is a solution to the homogeneous system \( Ax = 0 \). Is \( A \) singular or nonsingular? Justify your answer.

In Exercises 27 and 28, find the inverse of the given partitioned matrix \( A \) and express \( A^{-1} \) as a partitioned matrix.

27. \[
\begin{bmatrix} 5 & 2 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}
\]

28. \[
\begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 6 \\ 7 & 0 & 0 \end{bmatrix}
\]

In Exercises 29 and 30, find the inverse of the given bit matrices, if possible.

29. (a) \[
\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\]

(b) \[
\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
\]

(c) \[
\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

30. (a) \[
\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}
\]

(b) \[
\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}
\]

(c) \[
\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}
\]

In Exercises 31 and 32, determine which bit linear systems have a nontrivial solution.

31. (a) \( x + y + z = 0 \) \( \quad x + y + z = 0 \)

(b) \( x = 0 \) \( \quad x = 0 \)

32. (a) \( x + y + z = 0 \) \( \quad x + y + z = 0 \)

(b) \( y + z = 0 \) \( \quad y + z = 0 \)

32. (a) \( x + y + z = 0 \) \( \quad x + y + z = 0 \)

(b) \( y + z = 0 \) \( \quad y + z = 0 \)

Theoretical Exercises

T.1. Suppose that \( A \) and \( B \) are square matrices and \( AB = O \). If \( B \) is nonsingular, find \( A \).

T.2. Prove Corollary 1.2.

T.3. Let \( A \) be an \( n \times n \) matrix. Show that if \( A \) is singular, then the homogeneous system \( Ax = 0 \) has a nontrivial solution. (Hint: Use Theorem 1.12.)

T.4. Show that the matrix

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

is nonsingular if and only if \( ad - bc \neq 0 \). If this condition holds, show that

\[
A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.
\]

T.5. Show that the matrix

\[
\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}
\]

is nonsingular, and compute its inverse.
T.6. Show that the inverse of a nonsingular upper (lower) triangular matrix is upper (lower) triangular.

T.7. Show that if A is singular and Ax = b, b \neq 0 has one solution, then it has infinitely many. (Hint: Use Exercise T.13 in Section 1.6.)

T.8. Show that if A is a nonsingular symmetric matrix, then \( A^{-1} \) is symmetric.

T.9. Let \( A \) be a diagonal matrix with nonzero diagonal entries \( a_{11}, a_{22}, \ldots, a_{nn} \). Show that \( A^{-1} \) is nonsingular and that \( A^{-1} \) is a diagonal matrix with diagonal entries \( 1/a_{11}, 1/a_{22}, \ldots, 1/a_{nn} \).

T.10. If \( B = PAP^{-1} \), express \( B^2, B^3, \ldots, B^k \), where \( k \) is a positive integer, in terms of \( A, P \), and \( P^{-1} \).

T.11. Make a list of all possible \( 2 \times 2 \) bit matrices and then determine which are nonsingular. (See Exercise T.13 in Section 1.2.)

T.12. If \( A \) and \( B \) are nonsingular \( 3 \times 3 \) bit matrices, is it possible that \( AB = O \)? Explain.

T.13. Determine which \( 2 \times 2 \) bit matrices \( A \) have the property that \( A^2 = O \). (See Exercise T.13 in Section 1.2.)

MATLAB Exercises

In order to use MATLAB in this section, you should first have read Chapter 12 through Section 12.5.

ML.1. Using MATLAB, determine which of the following matrices are nonsingular. Use command \texttt{rref}.

\[
\text{(a) } A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \\
\text{(b) } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \\
\text{(c) } A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}
\]

ML.2. Using MATLAB, determine which of the following matrices are nonsingular. Use command \texttt{rref}.

\[
\text{(a) } A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \\
\text{(b) } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \\
\text{(c) } A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

ML.3. Using MATLAB, determine the inverse of each of the following matrices. Use command \texttt{rref([A eye(size(A))])}.

\[
\text{(a) } A = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \\
\text{(b) } A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}
\]

ML.4. Using MATLAB, determine the inverse of each of the following matrices. Use command \texttt{rref([A eye(size(A))])}.

\[
\text{(a) } A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
\text{(b) } A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}
\]

ML.5. Using MATLAB, determine a positive integer \( t \) so that \((tI - A)\) is singular.

\[
\text{(a) } A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \\
\text{(b) } A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 4 & 1 \\ 0 & 0 & -4 \end{bmatrix}
\]

ML.6. Determine which of the bit matrices in Exercises 29 and 30 have an inverse using \texttt{binreduce}.

ML.7. Determine which of the bit linear systems in Exercises 31 and 32 have a nontrivial solution using \texttt{binreduce}.

ML.8. Determine which of the following matrices has an inverse using \texttt{binreduce}.

\[
\text{(a) } A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
\text{(b) } A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}
\]

ML.9. Let \( B = \text{bigen}(1, 7, 3) \); that is, the matrix whose columns are the binary representations of the integers 1 through 7 using three bits. Determine two \( 3 \times 3 \) submatrices that have an inverse and two that do not.

1.8 LU-FACTORIZATION (OPTIONAL)

In this section we discuss a variant of Gaussian elimination (presented in Section 1.6) that decomposes a matrix as a product of a lower triangular matrix and an upper triangular matrix. This decomposition leads to an algorithm for solving a linear system \( Ax = b \) that is the most widely used method on computers for solving a linear system. A main reason for the popularity of this method is that it provides the cheapest way of solving a linear system.
1.8 Exercises

In Exercises 1 through 4, solve the linear system $Ax = b$ with the given LU-factorization of the coefficient matrix $A$. Solve the linear system using a forward substitution followed by a back substitution.

1. $A = \begin{bmatrix} 2 & 8 & 0 \\ 2 & 2 & -3 \\ 1 & 2 & 7 \end{bmatrix}, \quad b = \begin{bmatrix} 18 \\ 3 \\ 12 \end{bmatrix}$

2. $U = \begin{bmatrix} 2 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & -1 & 4 \end{bmatrix}$, \quad $L = \begin{bmatrix} 8 & 12 & -4 \\ 6 & 5 & 7 \\ 2 & 1 & 6 \end{bmatrix}$, \quad $b = \begin{bmatrix} 36 \\ 11 \\ 16 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & 3 & 0 & 1 \\ 4 & 5 & 3 & 3 \\ -2 & -6 & 7 & 7 \\ 8 & 9 & 5 & 21 \end{bmatrix}$, \quad $b = \begin{bmatrix} -2 \\ -2 \\ -16 \\ -66 \end{bmatrix}$

4. $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 3 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$, \quad $U = \begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix}$, \quad $b = \begin{bmatrix} 6 \\ 13 \\ -20 \\ -15 \end{bmatrix}$

5. $U = \begin{bmatrix} 4 & 2 & 1 & 0 \\ 0 & -4 & 2 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

In Exercises 5 through 10, find an LU-factorization of the coefficient matrix of the given linear system $Ax = b$. Solve the linear system using a forward substitution followed by a back substitution.

5. $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 10 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$.

6. $A = \begin{bmatrix} -3 & 1 & -2 \\ -12 & 10 & -6 \end{bmatrix}, \quad b = \begin{bmatrix} 15 \\ 82 \end{bmatrix}$

7. $A = \begin{bmatrix} 4 & 2 & 3 \\ 2 & 0 & 5 \\ 1 & 2 & 1 \end{bmatrix}$, \quad $b = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$

8. $A = \begin{bmatrix} -5 & 4 & 0 & 1 \\ -30 & 27 & 2 & 7 \\ -10 & 1 & -2 & 1 \end{bmatrix}$, \quad $b = \begin{bmatrix} -102 \\ -17 \end{bmatrix}$

9. $A = \begin{bmatrix} 2 & 1 & 0 & -4 \\ 1 & 0 & 0.25 & -1 \\ -2 & -1.1 & 0.25 & 6.2 \\ 4 & 2.2 & 0.3 & -2.4 \end{bmatrix}$, \quad $b = \begin{bmatrix} -3 \\ -1.5 \\ 5.6 \\ 2.2 \end{bmatrix}$

10. $A = \begin{bmatrix} 4 & 1 & 0.25 & -0.5 \\ 0.8 & 0.6 & 1.25 & -2.6 \\ -1.6 & -0.08 & 0.01 & 0.2 \\ 8 & 1.52 & -0.6 & -1.3 \end{bmatrix}$, \quad $b = \begin{bmatrix} -0.15 \\ 9.77 \\ 1.69 \\ -4.576 \end{bmatrix}$

MATLAB Exercises

Routine luper provides a step-by-step procedure in MATLAB for obtaining the LU-factorization discussed in this section. Once we have the LU-factorization, routines forsub and bksub can be used to perform the forward and back substitution, respectively. Use help for further information on these routines.

ML.1. Use luper in MATLAB to find an LU-factorization of $A = \begin{bmatrix} 2 & 8 & 0 \\ 2 & 2 & -3 \\ 1 & 2 & 7 \end{bmatrix}$.

ML.2. Use luper in MATLAB to find an LU-factorization of $A = \begin{bmatrix} 8 & -1 & 2 \\ 3 & 7 & 2 \\ 1 & 1 & 5 \end{bmatrix}$.

ML.3. Solve the linear system in Example 2 using luper, forsub, and bksub in MATLAB. Check your LU-factorization using Example 3.

ML.4. Solve Exercises 7 and 8 using luper, forsub, and bksub in MATLAB.
THE DETERMINANT OF BIT MATRICES (OPTIONAL)

The properties and techniques for the determinant developed in this section apply to bit matrices, where computations are carried out using binary arithmetic.

**EXAMPLE 20**

The determinant of the $2 \times 2$ bit matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

computed by using the technique developed in Example 5 is

$$\det(A) = (1)(1) - (1)(0) = 1.$$

**EXAMPLE 21**

The determinant of the $3 \times 3$ bit matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

computed by using the technique developed in Example 6 is

$$\det(A) = (1)(1)(1) + (0)(0)(0) + (1)(1)(1)
- (1)(0)(1) - (1)(0)(1) - (0)(1)(1)
= 1 + 0 + 1 - 0 - 0 - 0 = 1 + 1 = 0.$$

**EXAMPLE 22**

Use the computation via reduction to triangular form to evaluate the determinant of the bit matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

**Solution**

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

By Theorem 3.3, $\det(A) = 0$.

**Key Terms**

Permutation  
$n$ factorial  
Inversion  
Even permutation  
Odd permutation  
Determinant  
Computation via reduction to triangular form

**3.1 Exercises**

1. Find the number of inversions in each of the following permutations of $S = \{1, 2, 3, 4, 5\}$.
   (a) 52134  (b) 45213  (c) 42135  
   (d) 13542  (e) 35241  (f) 12345

2. Determine whether each of the following permutations of $S = \{1, 2, 3, 4\}$ is even or odd.
   (a) 4213  (b) 1243  (c) 1234  
   (d) 3214  (e) 1423  (f) 2431
3. Determine the sign associated with each of the following permutations of $S = \{1, 2, 3, 4, 5\}$.

(a) 25431 \hspace{1cm} (b) 31245 \hspace{1cm} (c) 21345

(d) 52341 \hspace{1cm} (e) 34125 \hspace{1cm} (f) 41253

4. In each of the following pairs of permutations of $S = \{1, 2, 3, 4, 5, 6\}$, verify that the number of inversions differs by an odd number.

(a) 436215 and 416235

(b) 623415 and 523416

(c) 321564 and 341562

(d) 123564 and 423561

5. In Exercises 5 and 6, evaluate the determinants using Equation (2).

5. (a) $\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}$  \hspace{1cm} (b) $\begin{vmatrix} 0 & 3 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -5 \end{vmatrix}$

(c) $\begin{vmatrix} 4 & 2 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 3 \end{vmatrix}$  \hspace{1cm} (d) $\begin{vmatrix} 4 & 2 & 2 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}$

6. (a) $\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix}$  \hspace{1cm} (b) $\begin{vmatrix} 0 & 0 & -2 \\ 0 & 3 & 0 \\ 4 & 0 & 0 \end{vmatrix}$

(c) $\begin{vmatrix} 3 & 4 & 2 \\ 2 & 5 & 0 \\ 3 & 0 & 0 \end{vmatrix}$  \hspace{1cm} (d) $\begin{vmatrix} -4 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 3 & 1 & 0 & 2 \\ 1 & 3 & 0 & 3 \end{vmatrix}$

7. Let $A = [a_{ij}]$ be a 4 × 4 matrix. Write the general expression for $\det(A)$ using Equation (2).

8. If

$|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -4,$

find the determinants of the following matrices:

$B = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix},$

$C = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 2c_1 & 2c_2 & 2c_3 \end{bmatrix},$

and

$D = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 + 4c_1 & b_2 + 4c_2 & b_3 + 4c_3 \end{bmatrix}.$

9. If

$|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 3,$

find the determinants of the following matrices:

$B = \begin{bmatrix} a_1 + 2b_1 - 3c_1 & a_2 + 2b_2 - 3c_2 & a_3 + 2b_3 - 3c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix},$

$C = \begin{bmatrix} a_1 & 3a_2 & a_3 \\ b_1 & 3b_2 & b_3 \\ c_1 & 3c_2 & c_3 \end{bmatrix},$

and

$D = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$

10. If

$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & 1 \\ 2 & 5 & 1 \end{bmatrix},$

verify that $\det(A) = \det(A^T)$.

11. Evaluate:

(a) $\det\left(\begin{bmatrix} \lambda - 1 & 2 \\ 3 & \lambda - 2 \end{bmatrix}\right)$

(b) $\det(\lambda I_2 - A)$, where $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

12. Evaluate:

(a) $\det\left(\begin{bmatrix} \lambda - 1 & -1 & -2 \\ 0 & \lambda - 2 & 2 \\ 0 & 0 & \lambda - 3 \end{bmatrix}\right)$

(b) $\det(\lambda I_3 - A)$, where $A = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

13. For each of the matrices in Exercise 11, find all values of $\lambda$ for which the determinant is 0.

14. For each of the matrices in Exercise 12, find all values of $\lambda$ for which the determinant is 0.

In Exercises 15 and 16, compute the indicated determinant.

15. (a) $\begin{vmatrix} 0 & 2 & -5 \\ 0 & 4 & 6 \\ 0 & 0 & -1 \end{vmatrix}$  \hspace{1cm} (b) $\begin{vmatrix} 6 & 6 & 3 & -2 \\ 0 & 4 & 7 & 5 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 2 \end{vmatrix}$

(c) $\begin{vmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{vmatrix}$

16. (a) $\begin{vmatrix} 6 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 5 & 9 & -3 & 0 \\ 4 & 1 & -3 & 2 \end{vmatrix}$  \hspace{1cm} (b) $\begin{vmatrix} 7 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -3 \end{vmatrix}$

(c) $\begin{vmatrix} 2 & 6 & -5 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{vmatrix}$
In Exercises 17 through 20, evaluate the given determinant via reduction to triangular form.

17. (a) \[
\begin{bmatrix}
4 & -3 & 5 \\
5 & 2 & 0 \\
2 & 0 & 4 \\
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
2 & 0 & 1 & 4 \\
0 & 3 & 2 & -4 \\
2 & 3 & -1 & 0 \\
11 & 8 & -4 & 6 \\
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
4 & 1 & 2 \\
0 & 2 & 3 \\
0 & 0 & -3 \\
\end{bmatrix}
\]

18. (a) \[
\begin{bmatrix}
-1 & 2 & 0 & 0 \\
1 & 2 & -3 & 0 \\
1 & 5 & 3 & 5 \\
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
4 & 1 & 3 \\
2 & 3 & 0 \\
1 & 3 & 2 \\
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 1 & 0 \\
-3 & 1 & 2 \\
\end{bmatrix}
\]

19. (a) \[
\begin{bmatrix}
3 & -2 & 1 & 5 \\
-2 & 0 & 1 & -3 \\
8 & -2 & 6 & 4 \\
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
1 & 3 & -4 \\
-2 & 1 & 2 \\
-9 & 15 & 0 \\
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
1 & 1 & 2 \\
0 & 2 & -2 \\
0 & 0 & 3 \\
\end{bmatrix}
\]

20. (a) \[
\begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
2 & 1 & 0 \\
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
2 & 0 & 0 & 0 \\
-5 & 3 & 0 & 0 \\
3 & 2 & 4 & 0 \\
4 & 2 & 1 & -5 \\
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
1 & 2 & -1 \\
3 & 2 & 0 \\
1 & 4 & 3 \\
\end{bmatrix}
\]

21. Verify that \(\det(AB) = \det(A) \det(B)\) for the following:

(a) \[
A = \begin{bmatrix}
-1 & 2 & 3 \\
-2 & 3 & 1 \\
0 & 1 & 0 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 & 2 \\
3 & -2 & 5 \\
2 & 1 & 3 \\
\end{bmatrix}
\]

(b) \[
A = \begin{bmatrix}
2 & 3 & 6 \\
0 & 3 & 2 \\
0 & 0 & -4 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
3 & 0 & 0 \\
4 & 5 & 0 \\
0 & 0 & 4 \\
\end{bmatrix}
\]

22. If \(|A| = -4\), find

(a) \(|A^2|\) \quad (b) \(|A^4|\) \quad (c) \(|A^{-1}|\)

23. If \(A\) and \(B\) are \(n \times n\) matrices with \(|A| = 2\) and \(|B| = -3\), calculate \(|A^{-1}B^T|\).

In Exercises 24 and 25, evaluate the given determinant of the bit matrices using techniques developed in Examples 5 and 6.

24. (a) \[
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
1 & 1 \\
0 & 1 \\
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]

25. (a) \[
\begin{bmatrix}
1 & 1 \\
1 & 0 \\
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
1 & 1 \\
0 & 0 \\
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
\end{bmatrix}
\]

In Exercises 26 and 27, evaluate the given determinant of the bit matrices via reduction to triangular form.

26. (a) \[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
1 & 0 & 1 \\
1 & 0 & 1 \\
0 & 1 & 0 \\
\end{bmatrix}
\]

27. (a) \[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 0 \\
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 0 & 0 \\
\end{bmatrix}
\]

Theoretical Exercises

T.1. Show that if we interchange two numbers in the permutation \(j_i j_2 \cdots j_n\), then the number of inversions is either increased or decreased by an odd number. (Hint: First show that if two adjacent numbers are interchanged, the number of inversions is either increased or decreased by 1. Then show that any interchange of any two numbers can be achieved by an odd number of successive interchanges of adjacent numbers.)

T.2. Prove Theorem 3.7 for the lower triangular case.

T.3. Show that if \(c\) is a real number and \(A\) is \(n \times n\), then \(\det(cA) = c^n \det(A)\).

T.4. Prove Corollary 3.2.

T.5. Show that if \(\det(AB) = 0\), then \(\det(A) = 0\) or \(\det(B) = 0\).

T.6. Is \(\det(AB) = \det(BA)\)? Justify your answer.

T.7. Show that if \(A\) is a matrix such that in each row and in each column one and only one element is \(\neq 0\), then \(\det(A) \neq 0\).

T.8. Show that if \(AB = I_n\), then \(\det(A) \neq 0\) and \(\det(B) \neq 0\).

T.9. (a) Show that if \(A = A^{-1}\), then \(\det(A) = \pm 1\).

(b) Show that if \(A^T = A^{-1}\), then \(\det(A) = \pm 1\).

T.10. Show that if \(A\) is a nonsingular matrix such that \(A^2 = A\), then \(\det(A) = 1\).
T.11. Show that
\[ \det(A^T B^T) = \det(A) \det(B^T) = \det(A^T) \det(B). \]

T.12. Show that
\[
\begin{vmatrix}
  a^2 & a & 1 \\
  b^2 & b & 1 \\
  c^2 & c & 1 \\
\end{vmatrix} = (b-a)(c-a)(b-c).
\]
This determinant is called a Vandermonde* determinant.

T.13. Let \( A = [a_{ij}] \) be an upper triangular matrix. Show that \( A \) is nonsingular if and only if \( a_{ii} \neq 0, 1 \leq i \leq n \).

T.14. Show that if \( \det(A) = 0 \), then \( \det(AB) = 0 \).

T.15. Show that if \( A^n = O \), for some positive integer \( n \), then \( \det(A) = 0 \).

T.16. Show that if \( A \) is \( n \times n \), with \( A \) skew symmetric \( (A^T = -A) \), see Section 1.4, Exercise T.24), and \( n \) odd, then \( \det(A) = 0 \).

T.17. Prove Corollary 3.1.

T.18. When is a diagonal matrix nonsingular? (Hint: See Exercise T.7.)

T.19. Using Exercise T.13 in Section 1.2, determine how many \( 2 \times 2 \) bit matrices have determinant 0 and how many have determinant 1.

---

**MATLAB Exercises**

In order to use MATLAB in this section, you should first have read Chapter 12 through Section 12.5.

**ML.1.** Use the routine `reduce` to perform row operations and keep track by hand of the changes in the determinant as in Example 17.

(a) \( A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix} \)

(b) \( A = \begin{bmatrix} 0 & 1 & 3 & -2 \\ -2 & 1 & 1 & 1 \\ 2 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \end{bmatrix} \)

**ML.2.** Use routine `reduce` to perform row operations and keep track by hand of the changes in the determinant as in Example 17.

(a) \( A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} \)

(b) \( A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix} \)

**ML.3.** MATLAB has command `det`, which returns the value of the determinant of a matrix. Just type `det(A)`. Find the determinant of each of the following matrices using `det`.

(a) \( A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \)

(b) \( A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix} \)

**ML.4.** Use `det` (see Exercise ML.3) to compute the determinant of each of the following.

(a) \( 5 \times \text{eye}(\text{size}(A)) - A \), where
\[
A = \begin{bmatrix} 2 & 3 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}
\]

(b) \( (3 \times \text{eye}(\text{size}(A)) - A)^*2 \), where
\[
A = \begin{bmatrix} 1 & 1 & 0 \\ 5 & 2 \end{bmatrix}
\]

(c) \( \text{invert}(A) \times A \), where
\[
A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}
\]

**ML.5.** Determine a positive integer \( t \) so that \( \det(t \times \text{eye}(\text{size}(A)) - A) = 0 \), where
\[
A = \begin{bmatrix} 5 & 2 \\ -1 & 2 \end{bmatrix}
\]

---

*Alexandre-Théophile Vandermonde (1735–1796) was born in Paris. His father, a physician, tried to steer him toward a musical career. His published mathematical work consisted of four papers that were presented over a two-year period. He is generally considered the founder of the theory of determinants and also developed formulas for solving general quadratic, cubic, and quartic equations. Vandermonde was a cofounder of the Conservatoire des Arts et Métiers and was its director from 1782. In 1795 he helped to develop a course in political economy. He was an active revolutionary in the French Revolution and was a member of the Commune of Paris and the club of the Jacobins.*
POLYNOMIAL INTERPOLATION REVISITED

At the end of Section 1.7 we discussed the problem of finding a quadratic polynomial that interpolates the points \((x_1, y_1), (x_2, y_2), (x_3, y_3), x_1 \neq x_2, x_1 \neq x_3, \text{ and } x_2 \neq x_3\). Thus, the polynomial has the form

\[
y = a_2x^2 + a_1x + a_0
\]

(this was Equation (15) in Section 1.6). Substituting the given points in (9), we obtain the linear system

\[
\begin{align*}
a_2x_1^2 + a_1x_1 + a_0 &= y_1 \\
a_2x_2^2 + a_1x_2 + a_0 &= y_2 \\
a_2x_3^2 + a_1x_3 + a_0 &= y_3.
\end{align*}
\]

(10)

The coefficient matrix of this linear system is

\[
\begin{bmatrix}
x_1^2 & x_1 & 1 \\
x_2^2 & x_2 & 1 \\
x_3^2 & x_3 & 1
\end{bmatrix}
\]

whose determinant is the Vandermonde determinant (see Exercise T.12 in Section 3.1), which has the value

\[(x_2 - x_1)(x_3 - x_1)(x_2 - x_3).
\]

Since the three given points are distinct, the Vandermonde determinant is not zero. Hence, the coefficient matrix of the linear system in (10) is nonsingular, which implies that the linear system has a unique solution. Thus there is a unique interpolating quadratic polynomial. The general proof for \(n\) points is similar.

OTHER APPLICATIONS OF DETERMINANTS

In Section 4.1 we use determinants to compute the area of a triangle and in Section 5.1 to compute the area of a parallelepiped.

---

Key Terms

Minor
Cofactor
Adjoint

3.2 Exercises

1. Let

\[
A = \begin{bmatrix}
1 & 0 & -2 \\
3 & 1 & 4 \\
5 & 2 & -3
\end{bmatrix}.
\]

Compute all the cofactors.

2. Let

\[
A = \begin{bmatrix}
1 & 0 & 3 & 0 \\
2 & 1 & 4 & -1 \\
3 & 2 & 4 & 0 \\
0 & 3 & -1 & 0
\end{bmatrix}.
\]

Compute all the cofactors of the elements in the second row and all the cofactors of the elements in the third column.
In Exercises 3 through 6, evaluate the determinants using Theorem 3.9.

3. (a) \[
\begin{vmatrix}
1 & 2 & 3 \\
-1 & 5 & 2 \\
3 & 2 & 0 \\
\end{vmatrix}
\]
(b) \[
\begin{vmatrix}
4 & -4 & 2 & 1 \\
1 & 2 & 0 & 3 \\
2 & 0 & 3 & 4 \\
0 & -3 & 2 & 1 \\
\end{vmatrix}
\]
(c) \[
\begin{vmatrix}
4 & -2 & 0 \\
0 & 2 & 4 \\
-1 & 1 & -3 \\
\end{vmatrix}
\]
4. (a) \[
\begin{vmatrix}
2 & 2 & -3 & 1 \\
0 & 1 & 2 & -1 \\
3 & -1 & 4 & 1 \\
2 & 3 & 0 & 0 \\
\end{vmatrix}
\]
(b) \[
\begin{vmatrix}
0 & 1 & -2 \\
-1 & 3 & 1 \\
2 & -2 & 3 \\
2 & 1 & -3 \\
\end{vmatrix}
\]
(c) \[
\begin{vmatrix}
0 & 1 & 2 \\
-4 & 2 & 1 \\
3 & 1 & 2 & -1 \\
2 & 0 & 3 & -7 \\
0 & -1 & 1 & -5 \\
\end{vmatrix}
\]
5. (a) \[
\begin{vmatrix}
3 & 1 & 0 \\
3 & 2 & 1 \\
3 & -3 & 0 \\
2 & 0 & 2 \\
2 & 1 & -3 \\
0 & 0 & -1 & 3 \\
0 & 1 & 2 & 1 \\
3 & 3 & 0 & 0 \\
\end{vmatrix}
\]
(b) \[
\begin{vmatrix}
2 & 0 & 1 \\
3 & 2 & -1 \\
1 & 2 & 4 \\
1 & -5 & 6 \\
1 & -5 & 6 \\
0 & 2 & 1 \\
0 & 1 & 0 \\
0 & 0 & 2 \\
\end{vmatrix}
\]
(c) \[
\begin{vmatrix}
4 & 2 & 0 \\
1 & 1 & 2 \\
-1 & 2 & -1 \\
3 & 2 & 1 \\
1 & 4 & 2 \\
\end{vmatrix}
\]
7. Verify Theorem 3.10 for the matrix
\[
A = \begin{bmatrix}
-2 & 3 & 0 \\
4 & 1 & -3 \\
2 & 0 & 1 \\
\end{bmatrix}
\]
by computing \(a_{11}A_{12} + a_{21}A_{22} + a_{31}A_{32}\).

8. Let
\[
A = \begin{bmatrix}
2 & 1 & 3 \\
-1 & 2 & 0 \\
3 & -2 & 1 \\
\end{bmatrix}
\]

(a) Find \(\text{adj} \ A\).
(b) Compute \(\text{det}(A)\).
(c) Verify Theorem 3.11; that is, show that
\[
A(\text{adj} \ A) = (\text{adj} \ A)A = \text{det}(A)I_3,
\]

9. Let
\[
A = \begin{bmatrix}
6 & 2 & 8 \\
-3 & 4 & 1 \\
4 & -4 & 5 \\
\end{bmatrix}
\]

(a) Find \(\text{adj} \ A\).
(b) Compute \(\text{det}(A)\).
(c) Verify Theorem 3.11; that is, show that
\[
A(\text{adj} \ A) = (\text{adj} \ A)A = \text{det}(A)I_3.
\]

In Exercises 10 through 13, compute the inverses of the matrices, if they exist, using Corollary 3.3.

10. (a) \[
\begin{bmatrix}
3 & 2 \\
-3 & 4 \\
\end{bmatrix}
\]
(b) \[
\begin{bmatrix}
4 & 2 \\
0 & 1 \\
\end{bmatrix}
\]
(c) \[
\begin{bmatrix}
2 & 0 & -1 \\
3 & 7 & 2 \\
1 & 1 & 0 \\
\end{bmatrix}
\]
11. (a) \[
\begin{bmatrix}
1 & 2 & -3 \\
-4 & -5 & 2 \\
-1 & 1 & 7 \\
\end{bmatrix}
\]
(b) \[
\begin{bmatrix}
2 & 3 \\
-1 & 2 \\
\end{bmatrix}
\]
(c) \[
\begin{bmatrix}
4 & 0 & 2 \\
0 & 3 & 4 \\
0 & 1 & -2 \\
\end{bmatrix}
\]
12. (a) \[
\begin{bmatrix}
2 & 0 & 1 \\
3 & 2 & -1 \\
1 & 2 & 4 \\
1 & -5 & 6 \\
3 & -1 & 2 \\
\end{bmatrix}
\]
(b) \[
\begin{bmatrix}
5 & -1 \\
2 & -1 \\
\end{bmatrix}
\]
(c) \[
\begin{bmatrix}
-3 & 1 \\
2 & 0 \\
\end{bmatrix}
\]
13. (a) \[
\begin{bmatrix}
-3 & 1 \\
2 & 0 \\
\end{bmatrix}
\]
(b) \[
\begin{bmatrix}
4 & 0 & 0 \\
0 & -3 & 0 \\
0 & 0 & 2 \\
\end{bmatrix}
\]
(c) \[
\begin{bmatrix}
2 & -1 & 3 \\
-2 & 1 & 5 \\
0 & 0 & 2 \\
\end{bmatrix}
\]
14. Use Theorem 3.12 to determine which of the following matrices are nonsingular.

(a) \[
\begin{bmatrix}
1 & 2 & 3 \\
0 & 1 & 2 \\
2 & -3 & 1 \\
\end{bmatrix}
\]
(b) \[
\begin{bmatrix}
1 & 2 \\
3 & 4 \\
\end{bmatrix}
\]
(c) \[
\begin{bmatrix}
1 & 3 & 2 \\
2 & 1 & 4 \\
1 & -7 & 2 \\
\end{bmatrix}
\]
15. Use Theorem 3.12 to determine which of the following matrices are nonsingular:

(a) \[
\begin{bmatrix}
4 & 3 & -5 \\
-2 & -1 & 3 \\
4 & 6 & -2
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
1 & 3 & -1 & 2 \\
2 & -6 & 4 & 1 \\
3 & 5 & -1 & 3 \\
4 & -6 & 5 & 2
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
2 & 2 & -4 \\
1 & 5 & 2 \\
3 & 7 & -2
\end{bmatrix}
\]

(d) \[
\begin{bmatrix}
0 & 1 & 2 \\
1 & 2 & 0 \\
1 & 3 & 4
\end{bmatrix}
\]

16. Find all values of \( \lambda \) for which

(a) \[
\det \begin{bmatrix}
\lambda - 2 & 2 \\
3 & \lambda - 3
\end{bmatrix} = 0
\]

(b) \( \det(\lambda I_3 - A) = 0 \), where
\[
A = \begin{bmatrix}
1 & 0 & -1 \\
2 & 0 & 1 \\
0 & 0 & -1
\end{bmatrix}
\]

17. Find all values of \( \lambda \) for which

(a) \[
\det \begin{bmatrix}
\lambda - 1 & -4 \\
0 & \lambda - 4
\end{bmatrix} = 0
\]

(b) \( \det(\lambda I_3 - A) = 0 \), where
\[
A = \begin{bmatrix}
-3 & -1 & -3 \\
0 & 3 & 0 \\
-2 & -1 & -2
\end{bmatrix}
\]

18. Use Corollary 3.4 to find whether the following homogeneous systems have nontrivial solutions.

(a) \[
\begin{align*}
x - 2y + z &= 0 \\
2x + 3y + z &= 0 \\
3x + y + 2z &= 0
\end{align*}
\]

(b) \[
\begin{align*}
x + 2y + w &= 0 \\
x + 2y + 3z &= 0 \\
z + 2w &= 0 \\
y + 2z - w &= 0
\end{align*}
\]

19. Repeat Exercise 18 for the following homogeneous systems.

(a) \[
\begin{align*}
x + 2y - z &= 0 \\
2x + y + 2z &= 0 \\
3x - y + z &= 0
\end{align*}
\]

(b) \[
\begin{align*}
x + y + 2z + w &= 0 \\
2x - y + z - w &= 0 \\
3x + y + 2z + 3w &= 0 \\
2x - y - z + w &= 0
\end{align*}
\]

In Exercises 20 through 23, if possible, solve the given linear system by Cramer's rule.

20. \[
\begin{align*}
2x + 4y + 6z &= 2 \\
x + 2z &= 0 \\
2x + 3y - z &= -5
\end{align*}
\]

21. \[
\begin{align*}
x + y + z - 2w &= -4 \\
2y + z + 3w &= 4 \\
2x + y - z + 2w &= 5 \\
x - y + w &= 4
\end{align*}
\]

22. \[
\begin{align*}
2x + y + z &= 6 \\
3x + 2y - 2z &= -2 \\
x + y + 2z &= 4
\end{align*}
\]

23. \[
\begin{align*}
2x + 3y + 7z &= 2 \\
-2x - 4z &= 0 \\
x + 2y + 4z &= 0
\end{align*}
\]

In Exercises 24 and 25, determine which of the following bit matrices are nonsingular using any of the techniques in the list of nonsingular equivalences.

24. (a) \[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}
\]

25. (a) \[
\begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}
\]

Theoretical Exercises

T.1. Show by a column (row) expansion that if \( A = [a_{ij}] \) is upper (lower) triangular, then \( \det(A) = a_{11}a_{22} \cdots a_{nn} \).

T.2. If \( A = [a_{ij}] \) is a \( 3 \times 3 \) matrix, develop the general expression for \( \det(A) \) by expanding (a) along the second column, and (b) along the third row. Compare these answers with those obtained for Example 6 in Section 3.1.

T.3. Show that if \( A \) is symmetric, then \( \text{adj} \ A \) is also symmetric.

T.4. Show that if \( A \) is a nonsingular upper triangular matrix, then \( A^{-1} \) is also upper triangular.

T.5. Show that
\[
A = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]
is nonsingular if and only if \( ad - bc \neq 0 \). If this condition is satisfied, use Corollary 3.3 to find \( A^{-1} \).
T.6. Using Corollary 3.3, find the inverse of

\[ A = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}. \]

[Hint: See Exercise T.12 in Section 3.1, where \( \det(A) \) was computed.]

T.7. Show that if \( A \) is singular, then \( \text{adj} \ A \) is singular.
[Hint: Show that if \( A \) is singular, then \( A \text{adj} \ A = O \).]

T.8. Show that if \( A \) is an \( n \times n \) matrix, then

\[ \det(\text{adj} \ A) = [\det(A)]^{n-1}. \]

T.9. Do Exercise T.10 in Section 1.6 using determinants.

**MATLAB Exercises**

ML.1. In MATLAB there is a routine `cofactor` that computes the \((i, j)\) cofactor of a matrix. For directions on using this routine, type `help cofactor`. Use `cofactor` to check your hand computations for the matrix \( A \) in Exercise 1.

ML.2. Use the `cofactor` routine (see Exercise ML.1) to compute the cofactor of the elements in the second row of

\[ A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & -1 & 3 \\ 3 & 2 & 1 \end{bmatrix}. \]

ML.3. Use the `cofactor` routine to evaluate the determinant of \( A \) using Theorem 3.9.

\[ A = \begin{bmatrix} 4 & 0 & -1 \\ -2 & 2 & -1 \\ 0 & 4 & -3 \end{bmatrix} \]

**3.3 DETERMINANTS FROM A COMPUTATIONAL POINT OF VIEW**

In this book we have, by now, developed two methods for solving a linear system of \( n \) equations in \( n \) unknowns: Gauss–Jordan reduction and Cramer's rule. We also have two methods for finding the inverse of a nonsingular matrix: the method involving determinants and the method discussed in Section 1.7. In this section we discuss criteria to be considered when selecting one or another of these methods.

Most sizable problems in linear algebra are solved on computers so that it is natural to compare two methods by estimating their computing time for the same problem. Since addition is so much faster than multiplication, the number of multiplications is often used as a basis of comparison for two numerical procedures.

Consider the linear system \( Ax = b \), where \( A \) is \( 25 \times 25 \). If we find \( x \) by Cramer's rule, we must first obtain \( \det(A) \). We can find \( \det(A) \) by cofactor expansion, say \( \det(A) = a_{11}A_{11} + a_{21}A_{21} + \cdots + a_{n1}A_{n1} \), where we have expanded \( \det(A) \) along the first column. Note that if each cofactor is available, we require 25 multiplications. Now each cofactor \( A_{ij} \) is plus or minus the determinant of a \( 24 \times 24 \) matrix, which can be expanded along a given row or column, requiring 24 multiplications. Thus the computation of \( \det(A) \) requires more than \( 25 \times 24 \times \cdots \times 2 \times 1 = 25! \) multiplications. Even if
This linear combination can be written as the matrix product (verify)

\[
\begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 1 \\
0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix} =
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}.
\]

which has the form of a linear system. Forming the corresponding augmented matrix and transforming it to reduced row echelon form, we obtain (verify)

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1
\end{bmatrix}.
\]

Hence the linear system is consistent with \( c_1 = 0, c_2 = 1, \) and \( c_3 = 1. \) Thus \( u \) is in span \( \{v_1, v_2, v_3\} \).

---

**Key Terms**
- Subspace
- Zero subspace
- Closure property
- Solution space
- Linear combination

---

**6.2 Exercises**

1. The set \( W \) consisting of all the points in \( \mathbb{R}^2 \) of the form \((x, x)\) is a straight line. Is \( W \) a subspace of \( \mathbb{R}^2 \)? Explain.

2. Let \( W \) be the set of all points in \( \mathbb{R}^3 \) that lie in the \( xy \)-plane. Is \( W \) a subspace of \( \mathbb{R}^3 \)? Explain.

3. Consider the circle in the \( xy \)-plane centered at the origin whose equation is \( x^2 + y^2 = 1 \). Let \( W \) be the set of all vectors whose tail is at the origin and whose head is a point inside or on the circle. Is \( W \) a subspace of \( \mathbb{R}^2 \)? Explain.

4. Consider the unit square shown in the accompanying figure. Let \( W \) be the set of all vectors of the form \( \begin{bmatrix} x \\ y \end{bmatrix} \), where \( 0 \leq x \leq 1, \ 0 \leq y \leq 1 \). That is, \( W \) is the set of all vectors whose tail is at the origin and whose head is a point inside or on the square. Is \( W \) a subspace of \( \mathbb{R}^2 \)? Explain.

5. Which of the following subsets of \( \mathbb{R}^3 \) are subspaces of \( \mathbb{R}^3 \)? The set of all vectors of the form
   (a) \((a, b, 2)\)
   (b) \((a, b, c), \) where \( c = a + b \)
   (c) \((a, b, c), \) where \( c > 0 \)
6. Which of the following subsets of $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$? The set of all vectors of the form
   (a) $\langle a, b, c \rangle$, where $a = c = 0$
   (b) $\langle a, b, c \rangle$, where $a = -c$
   (c) $\langle a, b, c \rangle$, where $b = 2a + 1$

7. Which of the following subsets of $\mathbb{R}^4$ are subspaces of $\mathbb{R}^4$? The set of all vectors of the form
   (a) $\langle a, b, c, d \rangle$, where $a - b = 2$
   (b) $\langle a, b, c, d \rangle$, where $c = a + 2b$ and $d = a - 3b$
   (c) $\langle a, b, c, d \rangle$, where $a = 0$ and $b = -d$

8. Which of the following subsets of $\mathbb{R}^4$ are subspaces of $\mathbb{R}^4$? The set of all vectors of the form
   (a) $\langle a, b, c, d \rangle$, where $a = b = 0$
   (b) $\langle a, b, c, d \rangle$, where $a = 1$, $b = 0$, and $c + d = 1$
   (c) $\langle a, b, c, d \rangle$, where $a > 0$ and $b < 0$

9. Which of the following subsets of $\mathbb{P}_2$ are subspaces?
   The set of all polynomials of the form
   (a) $a_2t^2 + a_1t + a_0$, where $a_0 = 0$
   (b) $a_2t^2 + a_1t + a_0$, where $a_0 = 2$
   (c) $a_2t^2 + a_1t + a_0$, where $a_2 + a_1 = a_0$

10. Which of the following subsets of $\mathbb{P}_2$ are subspaces?
    The set of all polynomials of the form
    (a) $a_2t^2 + a_1t + a_0$, where $a_1 = 0$ and $a_0 = 0$
    (b) $a_2t^2 + a_1t + a_0$, where $a_1 = 2a_0$
    (c) $a_2t^2 + a_1t + a_0$, where $a_2 + a_1 + a_0 = 2$

11. (a) Show that $\mathbb{P}_2$ is a subspace of $\mathbb{P}_3$.
    (b) Show that $\mathbb{P}_n$ is a subspace of $\mathbb{P}_{n+1}$.

12. Show that $\mathbb{P}_n$ is a subspace of $\mathbb{P}$.

13. Show that $P$ is a subspace of the vector space defined in Example 5 of Section 6.1.

14. Let $u = (1, 2, -3)$ and $v = (-2, 3, 0)$ be two vectors in $\mathbb{R}^3$ and let $W$ be the subset of $\mathbb{R}^3$ consisting of all vectors of the form $au + bv$, where $a$ and $b$ are any real numbers. Give an argument to show that $W$ is a subspace of $\mathbb{R}^3$.

15. Let $u = (2, 0, 3, -4)$ and $v = (4, 2, -5, 1)$ be two vectors in $\mathbb{R}^4$ and let $W$ be the subset of $\mathbb{R}^4$ consisting of all vectors of the form $au + bv$, where $a$ and $b$ are any real numbers. Give an argument to show that $W$ is a subspace of $\mathbb{R}^4$.

16. Which of the following subsets of the vector space $\mathbb{M}_{23}$ defined in Example 4 of Section 6.1 are subspaces? The set of all matrices of the form
    (a) $\begin{bmatrix} a & b & c \\ d & 0 & 0 \end{bmatrix}$, where $b = a + c$
    (b) $\begin{bmatrix} a & b & c \\ d & 0 & 0 \end{bmatrix}$, where $c > 0$
    (c) $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, where $a = -2c$ and $f = 2e + d$

17. Which of the following subsets of the vector space $\mathbb{M}_{23}$ defined in Example 4 of Section 6.1 are subspaces? The set of all matrices of the form
    (a) $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, where $a = 2c + 1$
    (b) $\begin{bmatrix} 0 & 1 & a \\ b & c & 0 \end{bmatrix}$
    (c) $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, where $a + c = 0$ and $b + d + f = 0$

18. Which of the following subsets of the vector space $\mathbb{M}_{n n}$ are subspaces?
    (a) The set of all $n \times n$ symmetric matrices
    (b) The set of all $n \times n$ nonsingular matrices
    (c) The set of all $n \times n$ diagonal matrices

19. Which of the following subsets of the vector space $\mathbb{M}_{n n}$ are subspaces?
    (a) The set of all $n \times n$ singular matrices
    (b) The set of all $n \times n$ upper triangular matrices
    (c) The set of all $n \times n$ matrices whose determinant is 1

20. (Calculus Required) Which of the following subsets are subspaces of the vector space $C(-\infty, \infty)$ defined in Example 8?
    (a) All nonnegative functions
    (b) All constant functions
    (c) All functions $f$ such that $f(0) = 0$
    (d) All functions $f$ such that $f(0) = 5$
    (e) All differentiable functions.

21. (Calculus Required) Which of the following subsets of the vector space $C(-\infty, \infty)$ defined in Example 8 are subspaces?
    (a) All integrable functions
    (b) All bounded functions
    (c) All functions that are integrable on $[a, b]$
    (d) All functions that are bounded on $[a, b]$

22. (Calculus Required) Consider the differential equation
    $$y'' - y' + 2y = 0.$$ 
    A solution is a real-valued function $f$ satisfying the equation. Let $V$ be the set of all solutions to the given differential equation; define $\oplus$ and $\circ$ as in Example 5 in Section 6.1. Show that $V$ is a subspace of the vector space of all real-valued functions defined on $(-\infty, \infty)$. (See also Section 9.2.)

23. Determine which of the following subsets of $\mathbb{R}^2$ are subspaces.
4. Determine which of the following subsets of \( R^2 \) are subspaces.

25. In each part, determine whether the given vector \( \mathbf{v} \) belongs to span \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \), where

\[
\mathbf{v}_1 = (1, 0, 0, 1), \quad \mathbf{v}_2 = (1, -1, 0, 0),
\]

and

\[
\mathbf{v}_3 = (0, 1, 2, 1).
\]

(a) \( \mathbf{v} = (-1, 4, 2, 2) \)  
(b) \( \mathbf{v} = (1, 2, 0, 1) \)  
(c) \( \mathbf{v} = (-1, 1, 4, 3) \)  
(d) \( \mathbf{v} = (0, 1, 1, 0) \)

26. Which of the following vectors are linear combinations of

\[
A_1 = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \\
A_3 = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}.
\]

(a) \( \begin{bmatrix} 5 \\ -1 \end{bmatrix} \)  
(b) \( \begin{bmatrix} -3 \\ 3 \end{bmatrix} \)  
(c) \( \begin{bmatrix} 3 \\ 3 \end{bmatrix} \)  
(d) \( \begin{bmatrix} 2 \\ 2 \end{bmatrix} \)

27. In each part, determine whether the given vector \( p(t) \) belongs to span \( \{ p_1(t), p_2(t), p_3(t) \} \), where

\[
p_1(t) = t^2 - t, \\
p_2(t) = t^2 - 2t + 1, \\
p_3(t) = -t^2 + 1.
\]

(a) \( p(t) = 3t^2 - 3t + 1 \)  
(b) \( p(t) = t^2 - t + 1 \)  
(c) \( p(t) = t + 1 \)  
(d) \( p(t) = 2t^2 - t - 1 \)

Exercises 28 through 33 use bit matrices.

28. Let \( V = B^3 \). Determine if

\[
W = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\]

is a subspace of \( V \).

29. Let \( V = B^3 \). Determine if

\[
W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

is a subspace of \( V \).

30. Let \( V = B^4 \). Determine if \( W \), the set of all vectors in \( V \) with first entry zero, is a subspace of \( V \).

31. Let \( V = B^4 \). Determine if \( W \), the set of all vectors in \( V \) with second entry one, is a subspace of \( V \).

32. Let

\[
S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.
\]

Determine if \( \mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \) belongs to span \( S \).

33. Let

\[
S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.
\]

Determine if \( \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) belongs to span \( S \).
Theoretical Exercises

T.1. Prove Theorem 6.2.

T.2. Show that a subset $W$ of a vector space $V$ is a subspace of $V$ if and only if the following condition holds: If $u$ and $v$ are any vectors in $W$ and $a$ and $b$ are any scalars, then $au + bv$ is in $W$.

T.3. Show that the set of all solutions to $Ax = b$, where $A$ is $m \times n$, is not a subspace of $R^n$ if $b \neq 0$.

T.4. Prove Theorem 6.3.

T.5. Let $S = \{v_1, v_2, \ldots, v_k\}$ be a set of vectors in a vector space $V$, and let $W$ be a subspace of $V$ containing $S$. Show that $W$ contains $\text{span} \ S$.

T.6. If $A$ is a nonsingular matrix, what is the null space of $A$? Justify your answer.

T.7. Let $x_0$ be a fixed vector in a vector space $V$. Show that the set $W$ consisting of all scalar multiples $cx_0$ of $x_0$ is a subspace of $V$.

T.8. Let $A$ be an $m \times n$ matrix. Is the set $W$ of all vectors $x$ in $R^n$ such that $Ax = 0$ a subspace of $R^n$? Justify your answer.

T.9. Show that the only subspaces of $R^1$ are $\{0\}$ and $R^1$ itself.

T.10. Let $W_1$ and $W_2$ be subspaces of a vector space $V$. Let $W_1 + W_2$ be the set of all vectors $v$ in $V$ such that $v = w_1 + w_2$, where $w_1$ is in $W_1$ and $w_2$ is in $W_2$. Show that $W_1 + W_2$ is a subspace of $V$.

T.11. Let $W_1$ and $W_2$ be subspaces of a vector space $V$ with $W_1 \cap W_2 = \{0\}$. Let $W_1 + W_2$ be as defined in Exercise T.10. Suppose that $V = W_1 + W_2$. Show that every vector in $V$ can be uniquely written as $w_1 + w_2$, where $w_1$ is in $W_1$ and $w_2$ is in $W_2$. In this case we write $V = W_1 \oplus W_2$ and say that $V$ is the direct sum of the subspaces $W_1$ and $W_2$.

T.12. Show that the set of all points in the plane $ax + by + cz = 0$ is a subspace of $R^3$.

T.13. Show that if a subset $W$ of a vector space $V$ does not contain the zero vector, then $W$ is not a subspace of $V$.

T.14. Let $V = B^3$ and $W = \{w_1\}$, where $w_1$ is any vector in $B^3$. Is $W$ a subspace of $V$?

T.15. Let $V = B^3$. Determine if there is a subspace of $V$ that contains exactly three different vectors.

T.16. In Example 14, $W$ is a subspace of $B^3$ with exactly four vectors. Determine two other subspaces of $B^3$ that contain exactly four vectors.

T.17. Determine all the subspaces of $B^3$ that contain $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

MATLAB Exercises

ML.1. Let $V$ be $R^3$ and let $W$ be the subset of $V$ of vectors of the form $(2, a, b)$, where $a$ and $b$ are any real numbers. Is $W$ a subspace of $V$? Use the following MATLAB commands to help you determine the answer.

\begin{verbatim}
a1 = fix(10 * randn);
a2 = fix(10 * randn);
b1 = fix(10 * randn);
b2 = fix(10 * randn);
v = [2 a1 b1]
w = [2 a2 b2]
v + w
3 * v
\end{verbatim}

ML.2. Let $V$ be $P_2$ and let $W$ be the subset of $V$ of vectors of the form $ax^2 + bx + 5$, where $a$ and $b$ are arbitrary real numbers. With each such polynomial in $W$ we associate a vector $(a, b, 5)$ in $R^3$. Construct commands like those in Exercise ML.1 to show that $W$ is not a subspace of $V$.

Before solving the following MATLAB exercises, you should have read Section 12.7.

ML.3. Use MATLAB to determine if vector $v$ is a linear combination of the members of set $S$.

(a) $S = \{v_1, v_2, v_3\}$
\begin{equation*}
\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}
\end{equation*}
\begin{equation*}
v = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}
\end{equation*}

(b) $S = \{v_1, v_2, v_3\}$
\begin{equation*}
\begin{pmatrix} 1 \\ 2 \\ -1 \\ -8 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}
\end{equation*}
\begin{equation*}
v = \begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix}
\end{equation*}

ML.4. Use MATLAB to determine if $v$ is a linear combination of the members of set $S$. If it is, express $v$ in terms of the members of $S$.

(a) $S = \{v_1, v_2, v_3\}$
\begin{equation*}
\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}
\end{equation*}
\begin{equation*}
v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\end{equation*}

(b) $S = \{A_1, A_2, A_3\}$
\begin{equation*}
\begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -3 \\ 1 \end{pmatrix}
\end{equation*}
\begin{equation*}
v = I_2
\end{equation*}
**ML.5.** Use MATLAB to determine if \( v \) is a linear combination of the members of set \( S \). If it is, express \( v \) in terms of the members of \( S \).

(a) \( S = \{ v_1, v_2, v_3, v_4 \} \)

\[
\begin{bmatrix}
1 & 0 & 2 & -2 \\
2 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 \\
0 & -1 & 0 & 1
\end{bmatrix}
\]

\( v = \begin{bmatrix} 0 \\ -1 \\ -2 \\ 1 \end{bmatrix} \)

(b) \( S = \{ p_1(t), p_2(t), p_3(t) \} \)

\[
= \{ 2t^2 - t + 1, t^2 - 2, t - 1 \}
\]

\( v = p(t) = 4t^2 + t - 5 \)

**ML.6.** In each part, determine whether \( v \) belongs to span \( S \), where

\[
S = \{ v_1, v_2, v_3 \} = \{(1, 1, 0, 1), (1, -1, 0, 1), (0, 1, 2, 1)\}.
\]

(a) \( v = (2, 3, 2, 3) \)

(b) \( v = (2, -3, -2, 3) \)

(c) \( v = (0, 1, 2, 3) \)

**ML.7.** In each part, determine whether \( p(t) \) belongs to span \( S \), where

\[
S = \{ p_1(t), p_2(t), p_3(t) \} = \{ t - 1, t + 1, t^2 + t + 1 \}.
\]

(a) \( p(t) = t^2 + 2t + 4 \)

(b) \( p(t) = 2t^2 + t - 2 \)

(c) \( p(t) = -2t^2 + 1 \)

### 6.3 Linear Independence

Thus far we have defined a mathematical system called a real vector space and noted some of its properties. We further observe that the only real vector space having a finite number of vectors in it is the vector space whose only vector is \( \mathbf{0} \), for if \( v \neq \mathbf{0} \) is in a vector space \( V \), then by Exercise T.4 in Section 6.1, \( cv \neq c'v \), where \( c \) and \( c' \) are distinct real numbers, and so \( V \) has infinitely many vectors in it. However, in this section and the following one we show that most vector spaces \( V \) studied here have a set composed of a finite number of vectors that completely describe \( V \); that is, we can write every vector in \( V \) as a linear combination of the vectors in this set. It should be noted that, in general, there is more than one such set describing \( V \). We now turn to a formulation of these ideas.

The vectors \( v_1, v_2, \ldots, v_k \) in a vector space \( V \) are said to span \( V \) if every vector in \( V \) is a linear combination of \( v_1, v_2, \ldots, v_k \). Moreover, if \( S = \{ v_1, v_2, \ldots, v_k \} \), then we also say that the set \( S \) spans \( V \), or that \( \{ v_1, v_2, \ldots, v_k \} \) spans \( V \), or that \( V \) is spanned by \( S \), or in the language of Section 6.2, span \( S = V \).

The procedure to check if the vectors \( v_1, v_2, \ldots, v_k \) span the vector space \( V \) is as follows.

**Step 1.** Choose an arbitrary vector \( v \) in \( V \).

**Step 2.** Determine if \( v \) is a linear combination of the given vectors. If it is, then the given vectors span \( V \). If it is not, they do not span \( V \).

Again, in Step 2, we investigate the consistency of a linear system, but this time for a right side that represents an arbitrary vector in a vector space \( V \).
6.3 Exercises

1. Which of the following vectors span \( \mathbb{R}^2 \)?
   (a) \((1, 2), (-1, 1)\)
   (b) \((0, 0), (1, 1), (-2, -2)\)
   (c) \((1, 3), (2, -3), (0, 2)\)
   (d) \((2, 4), (-1, 2)\)

2. Which of the following sets of vectors span \( \mathbb{R}^3 \)?
   (a) \[
   \{(1, -1, 2), (0, 1, 1)\}
   \]
   (b) \[
   \{(1, 2, -1), (6, 3, 0), (4, -1, 2), (2, -5, 4)\}
   \]
   (c) \[
   \{(2, 2, 3), (-1, -2, 1), (0, 1, 0)\}
   \]
   (d) \[
   \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}
   \]

3. Which of the following vectors span \( \mathbb{R}^4 \)?
   (a) \((1, 0, 0, 1), (0, 1, 0, 0), (1, 1, 1, 0), (0, 1, 1, 0)\)
   (b) \((1, 2, 1, 0), (1, 1, -1, 0), (0, 0, 0, 1)\)
   (c) \[
   \{(6, 4, -2, -4), (2, 0, 0, 1), (3, 2, -1, 2), (5, 6, -5, 2), (0, 4, -2, -1)\}
   \]
   (d) \[
   \{(1, 1, 0, 0), (1, 2, -1, 1), (0, 0, 1, 1), (2, 1, 2, 1)\}
   \]

4. Which of the following sets of polynomials span \( \mathbb{P}_3 \)?
   (a) \[
   \{t^3 + 1, t^2 + t, t + 1\}
   \]
   (b) \[
   \{t^3 + 1, t - 1, t^2 + t\}
   \]
   (c) \[
   \{t^2 + 2, 2t^2 - t + 1, t + 2, t^2 + t + 4\}
   \]
   (d) \[
   \{t^3 + 2t - 1, t^2 - 1\}
   \]

5. Do the polynomials \( t^3 + 2t + 1, t^2 + t + 2, t^3 + 2, t^3 + t - 5t + 2 \) span \( \mathbb{P}_3 \)?

6. Find a set of vectors spanning the solution space of \( Ax = 0 \), where
   \[
   A = \begin{bmatrix}
   1 & 0 & 1 & 0 \\
   1 & 2 & 3 & 1 \\
   2 & 1 & 3 & 1 \\
   1 & 2 & 1 & 1
   \end{bmatrix}.
   \]

7. Find a set of vectors spanning the null space of
   \[
   A = \begin{bmatrix}
   1 & 1 & 2 & -1 \\
   2 & 3 & 6 & -2 \\
   -2 & 1 & 2 & 2 \\
   0 & -2 & -4 & 0
   \end{bmatrix}.
   \]

8. Let
   \[
   x_1 = \begin{bmatrix}
   2 \\
   -1 \\
   1
   \end{bmatrix},
   x_2 = \begin{bmatrix}
   4 \\
   -7 \\
   -1
   \end{bmatrix},
   x_3 = \begin{bmatrix}
   1 \\
   2 \\
   2
   \end{bmatrix}
   \]

   belong to the solution space of \( Ax = 0 \). Is \( \{x_1, x_2, x_3\} \) linearly independent?

9. Let
   \[
   x_1 = \begin{bmatrix}
   1 \\
   2 \\
   0 \\
   1
   \end{bmatrix},
   x_2 = \begin{bmatrix}
   1 \\
   0 \\
   -1 \\
   1
   \end{bmatrix},
   x_3 = \begin{bmatrix}
   1 \\
   6 \\
   2 \\
   0
   \end{bmatrix}
   \]

   belong to the null space of \( A \). Is \( \{x_1, x_2, x_3\} \) linearly independent?

10. Which of the following sets of vectors in \( \mathbb{R}^3 \) are linearly dependent? For those that are, express one vector as a linear combination of the rest.
   (a) \( \{(1, 2, -1), (3, 2, 5)\} \)
   (b) \( \{(4, 2, 1), (2, 6, -5), (1, -2, 3)\} \)
   (c) \( \{(1, 0, 0), (0, 2, 3), (1, 2, 3), (3, 6, 0)\} \)
   (d) \( \{(1, 2, 3), (1, 1, 1), (1, 0, 1)\} \)

11. Consider the vector space \( \mathbb{R}^4 \). Follow the directions of Exercise 10.
   (a) \( \{(1, 1, 2, 1), (1, 0, 0, 2), (4, 6, 8, 6), (0, 3, 2, 1)\} \)
   (b) \( \{(1, -2, 3, -1), (-2, 4, -6, 2)\} \)
   (c) \( \{(1, 1, 1, 1), (2, 3, 1, 2), (3, 1, 2, 1), (2, 2, 1, 1)\} \)
   (d) \( \{(4, 2, -1, 3), (6, 5, -5, 1), (2, -1, 3, 5)\} \)

12. Consider the vector space \( \mathbb{P}_2 \). Follow the directions of Exercise 10.
   (a) \( \{t^2 + 1, t - 2, t + 3\} \)
   (b) \( \{2t^2 + 1, t^2 + 3, t\} \)
   (c) \( \{3t + 1, 3t^2 + 1, 2t^2 + t + 1\} \)
   (d) \( \{t^2 - 4, 5t^2 - 5t - 6, 3t^2 - 5t + 2\} \)

13. Consider the vector space \( M_{22} \). Follow the directions of Exercise 10.
   (a) \( \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}, \frac{3}{2} \begin{bmatrix} 2 & 6 \\ 4 & 6 \end{bmatrix} \)
   (b) \( \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \)
   (c) \( \frac{1}{2} \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} \)

14. Let \( V \) be the vector space of all real-valued continuous functions. Follow the directions of Exercise 10.
   (a) \( \{\cos t, \sin t, e^t\} \)
   (b) \( \{t, e^t, \sin t\} \)
   (c) \( \{t^2, t, e^t\} \)
   (d) \( \{\cos^2 t, \sin^2 t, \cos 2t\} \)

15. For what values of \( c \) are the vectors \( (-1, 0, -1), (2, 1, 2), (1, 1, c) \) in \( \mathbb{R}^3 \) linearly dependent?

16. For what values of \( \lambda \) are the vectors \( t + 3 \) and \( 2t + \lambda^2 + 2 \) in \( \mathbb{P}_1 \) linearly dependent?

17. Determine if the vectors \( \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \) and \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) span \( \mathbb{P}_1 \).

18. Determine if the vectors \( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) span \( \mathbb{P}_1 \).
19. Determine if the vectors \[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}, \text{ and } \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]
span \(R^4\).

20. Determine if the vectors in Exercise 17 are linearly independent.

21. Determine if the vectors in Exercise 19 are linearly independent.

22. Show that \(v_1, v_2, \text{ and } v_3\) in Example 15 are linearly dependent using Theorem 6.4.

**Theoretical Exercises**

T.1. Show that the vectors \(e_1, e_2, \ldots, e_n\) in \(R^n\) are linearly independent.

T.2. Let \(S_1\) and \(S_2\) be finite subsets of a vector space and let \(S_1\) be a subset of \(S_2\). Show that:
   (a) If \(S_1\) is linearly dependent, so is \(S_2\).
   (b) If \(S_2\) is linearly independent, so is \(S_1\).

T.3. Let \(S = \{v_1, v_2, \ldots, v_k\}\) be a set of vectors in a vector space. Show that \(S\) is linearly dependent if and only if one of the vectors in \(S\) is a linear combination of all the other vectors in \(S\).

T.4. Suppose that \(S = \{v_1, v_2, v_3\}\) is a linearly independent set of vectors in a vector space \(V\). Show that \(T = \{w_1, w_2, w_3\}\) is also linearly independent, where \(w_1 = v_1 + v_2 + v_3\), \(w_2 = v_1 + v_3\), and \(w_3 = v_1\).

T.5. Suppose that \(S = \{v_1, v_2, v_3\}\) is a linearly independent set of vectors in a vector space \(V\). Is \(T = \{w_1, w_2, w_3\}\), where \(w_1 = v_1 + v_2\), \(w_2 = v_1 + v_3\), \(w_3 = v_1 + v_3\), linearly dependent or linearly independent? Justify your answer.

T.6. Suppose that \(S = \{v_1, v_2, v_3\}\) is a linearly dependent set of vectors in a vector space \(V\). Is \(T = \{w_1, w_2, w_3\}\), where \(w_1 = v_1 + v_2\), \(w_2 = v_1 + v_3\), \(w_3 = v_1 + v_3\), linearly dependent or linearly independent? Justify your answer.

T.7. Let \(v_1, v_2, \text{ and } v_3\) be vectors in a vector space such that \(\{v_1, v_2\}\) is linearly independent. Show that if \(v_3\) does not belong to span \(\{v_1, v_2\}\), then \(\{v_1, v_2, v_3\}\) is linearly independent.

T.8. Let \(A\) be an \(m \times n\) matrix in reduced row echelon form. Show that the nonzero rows of \(A\), viewed as vectors in \(R^n\), form a linearly independent set of vectors.

T.9. Let \(S = \{u_1, u_2, \ldots, u_n\}\) be a set of vectors in a vector space, and let \(T = \{v_1, v_2, \ldots, v_m\}\), where each \(v_i\), \(i = 1, 2, \ldots, m\), is a linear combination of the vectors in \(S\). Show that \(w = b_1v_1 + b_2v_2 + \ldots + b_nv_n\) is a linear combination of the vectors in \(S\).

T.10. Suppose that \(\{v_1, v_2, \ldots, v_n\}\) is a linearly independent set of vectors in \(R^n\). Show that if \(A\) is an \(n \times n\) nonsingular matrix, then \([Av_1, Av_2, \ldots, Av_n]\) is linearly independent.

T.11. Let \(S_1\) and \(S_2\) be finite subsets of a vector space \(V\) and let \(S_1\) be a subset of \(S_2\). If \(S_2\) is linearly dependent, show by examples that \(S_1\) may be either linearly dependent or linearly independent.

T.12. Let \(S_1\) and \(S_2\) be finite subsets of a vector space and let \(S_1\) be a subset of \(S_2\). If \(S_1\) is linearly independent, show by examples that \(S_2\) may be either linearly dependent or linearly independent.

T.13. Let \(u\) and \(v\) be nonzero vectors in a vector space \(V\). Show that \([u, v]\) is linearly dependent if and only if there is a scalar \(k\) such that \(v = ku\). Equivalently, \([u, v]\) is linearly independent if and only if one of the vectors is not a multiple of the other.

T.14. (Uses material from Section 5.1) Let \(u\) and \(v\) be linearly independent vectors in \(R^3\). Show that \(u\) and \(u \times v\) form a basis for \(R^3\). [Hint: Form Equator (1) and take the dot product with \(u \times v\).]

T.15. Let \(W\) be a subspace of \(V\) spanned by the vectors \(w_1, w_2, \ldots, w_k\). Is there any vector \(v\) in \(V\) such that the span of \(\{w_1, w_2, \ldots, w_k, v\}\) will also be \(W\)? Describe all such vectors \(v\).

**MATLAB Exercises**

ML.1. Determine if \(S\) is linearly independent or linearly dependent.

(a) \(S = \{(1, 0, 0, 1), (0, 1, 1, 0), (1, 1, 1, 1)\}\)

(b) \(S = \{(1, 2), (2, -1), (3, 1)\}\)

(c) \(S = \{(2, 1, 2), (-2, 1, 1)\}\)
ML.2. Find a spanning set of the solution space of \( Ax = 0 \),
where
\[
A = \begin{bmatrix}
1 & 2 & 0 & 1 \\
1 & 1 & 1 & 2 \\
2 & -1 & 5 & 7 \\
0 & 2 & -2 & -2
\end{bmatrix}.
\]

## 6.4 BASIS AND DIMENSION

In this section we continue our study of the structure of a vector space \( V \) by determining a smallest set of vectors in \( V \) that completely describes \( V \).

### BASIS

#### DEFINITION

The vectors \( v_1, v_2, \ldots, v_k \) in a vector space \( V \) are said to form a **basis** for \( V \) if (a) \( v_1, v_2, \ldots, v_k \) span \( V \) and (b) \( v_1, v_2, \ldots, v_k \) are linearly independent.

#### Remark

If \( v_1, v_2, \ldots, v_k \) form a basis for a vector space \( V \), then they must be nonzero (see Example 12 in Section 6.3) and distinct and so we write them as a set \( \{v_1, v_2, \ldots, v_k\} \).

#### EXAMPLE 1

The vectors \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \) form a basis for \( R^2 \), the vectors \( e_1, e_2, \) and \( e_3 \) form a basis for \( R^3 \) and, in general, the vectors \( e_1, e_2, \ldots, e_n \) form a basis for \( R^n \). Each of these sets of vectors is called the **natural basis** or **standard basis** for \( R^2, R^3, \) and \( R^n \), respectively.

#### EXAMPLE 2

Show that the set \( S = \{v_1, v_2, v_3, v_4\} \), where \( v_1 = (1, 0, 1, 0), v_2 = (0, 1, -1, 2), v_3 = (0, 2, 2, 1), \) and \( v_4 = (1, 0, 0, 1) \) is a basis for \( R^4 \).

**Solution**

To show that \( S \) is linearly independent, we form the equation

\[
c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0
\]

and solve for \( c_1, c_2, c_3, \) and \( c_4 \). Substituting for \( v_1, v_2, v_3, \) and \( v_4 \), we obtain the linear system (verify)

\[
\begin{align*}
c_1 + c_4 &= 0 \\
c_2 + 2c_3 &= 0 \\
c_1 - c_2 + 2c_3 &= 0 \\
2c_2 + c_3 + c_4 &= 0,
\end{align*}
\]

which has as its only solution \( c_1 = c_2 = c_3 = c_4 = 0 \) (verify), showing that \( S \) is linearly independent. Observe that the coefficient matrix of the preceding linear system consists of the vectors \( v_1, v_2, v_3, \) and \( v_4 \) written in column form.

To show that \( S \) spans \( R^4 \), we let \( v = (a, b, c, d) \) be any vector in \( R^4 \). We now seek constants \( k_1, k_2, k_3, \) and \( k_4 \) such that

\[
k_1v_1 + k_2v_2 + k_3v_3 + k_4v_4 = v.
\]

Substituting for \( v_1, v_2, v_3, v_4, \) and \( v \), we find a solution (verify) for \( k_1, k_2, k_3, \) and \( k_4 \) to the resulting linear system for any \( a, b, c, \) and \( d \). Hence \( S \) spans \( R^4 \) and is a basis for \( R^4 \).
We form the augmented matrix and use row operations: Add row 1 to row 4, add row 2 to row 3, and add row 2 to row 4, to obtain the equivalent augmented matrix (verify)

\[
\begin{bmatrix}
1 & 0 & 1 & 0 & a \\
0 & 1 & 1 & 0 & b \\
0 & 0 & 1 & 1 & b + c \\
0 & 0 & 0 & 1 & a + b + d
\end{bmatrix}
\]

It follows that this system is inconsistent if the choice of bits $a$, $b$, and $d$ is such that $a + b + d \neq 0$. For example, if

\[
w = \begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix},
\]

then the system is inconsistent; hence $S$ does not span $B^4$ and is not a basis for $B^4$.

---

**Key Terms**

- **Basis**
  - Natural (or standard) basis
  - Finite-dimensional vector space

- **Infinite-dimensional vector space**
  - Dimension

---

**6.4 Exercises**

1. Which of the following sets of vectors are bases for $R^3$?
   (a) \{(1, 3), (1, -1)\}
   (b) \{(0, 0), (1, 2), (2, 4)\}
   (c) \{(1, 2), (2, -3), (3, 2)\}
   (d) \{(1, 3), (-2, 6)\}

2. Which of the following sets of vectors are bases for $R^3$?
   (a) \{(1, 1, 1), (0, 1, 0), (0, 1, -1)\}
   (b) \{(3, 2, 2), (-1, 2, 1), (0, 1, 0)\}
   (c) \{(1, 0, 0), (0, 2, 1), (3, 4, 1), (0, 1, 0)\}
   (d) \{(1, 0, 0), (0, 1, 0), (1, 1, 1), (0, 1, 1)\}

3. Which of the following sets of vectors are bases for $R^4$?
   (a) \{(-1, 2, 3, 4), (0, 1, 2, 0), (-1, 2, 3, 2), (0, 1, 0, 4)\}
   (b) \{(-2, 4, 6, 4), (0, 1, 2, 0), (1, 2, 3, 2), (-2, -1, 0, 4)\}
   (c) \{(0, 0, 1, 1), (1, 1, 1, 2), (1, 1, 0, 0), (2, 1, 2, 1)\}
   (d) \{-t^2 + t + 2, 2t^3 + 3t^2 + 3t + 3\}

4. Which of the following sets of vectors are bases for $P_2$?
   (a) \{x^2 + t + 2, 2t^2 + 2t + 3, 4t^3 - 1\}
   (b) \{t^2 + 2t, 2t^2 + 3t - 2\}
   (c) \{t^2 + 1, 3t^2 + 2t, 3t^2 + 2t + 1, 6t^2 + 6t + 3\}

5. Which of the following sets of vectors are bases for $P_3$?
   (a) \{t^3 + 2t^2 + 3t, 2t^3 + 1, 6t^2 + 8t + 4, 2t^2 + 3t + 1\}
   (b) \{t^3 - t + 1, t^3 - t, t^3 + t^2 + t\}
   (c) \{t^3 + 2t^2 + t + 1, t^3 + 2t^2 + t + 1, 2t^2 + 3t + 2, t^2 + t + 2\}
   (d) \{t^3 - t, t^3 + t^2 + 1, t - 1\}

6. Show that the matrices

\[
\begin{bmatrix}
1 & 1 \\
0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 \\
1 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix},
\begin{bmatrix}
0 & 1 \\
1 & 1
\end{bmatrix}
\]

form a basis for the vector space $M_{22}$.

In Exercises 7 and 8, determine which of the given subsets forms a basis for $R^3$. Express the vector $(2, 1, 3)$ as a linear combination of the vectors in each subset that is a basis.

7. (a) \{(1, 1, 1), (1, 2, 3), (0, 1, 0)\}
   (b) \{(1, 2, 3), (2, 1, 3), (0, 0, 0)\}

8. (a) \{(2, 1, 3), (1, 2, 1), (1, 1, 4), (1, 5, 1)\}
   (b) \{(1, 1, 2), (2, 2, 0), (3, 4, -1)\}
In Exercises 9 and 10, determine which of the given subsets forms a basis for \( P_2 \). Express \( 5t^2 - 3t + 8 \) as a linear combination of the vectors in each subset that is a basis.

9. (a) \([t^2 + 1, t - 1, t + 1]\)
   (b) \([t^2 + 1, t - 1]\)

10. (a) \([t^2 + 1, t^2 + 1]\)
    (b) \([t^2 + 1, t^2 - t + 1]\)

11. Let \( S = \{v_1, v_2, v_3, v_4\} \), where
    \[v_1 = (1, 2, 2), \quad v_2 = (3, 2, 1), \]
    \[v_3 = (11, 10, 7), \quad v_4 = (7, 6, 4).\]

Find a basis for the subspace \( W = \text{span} \ S \) of \( \mathbb{R}^3 \). What is \( \dim W \)?

12. Let \( S = \{v_1, v_2, v_3, v_4, v_5\} \), where
    \[v_1 = (1, 1, 0, -1), \quad v_2 = (0, 1, 2, 1), \]
    \[v_3 = (1, 0, 1, 1), \quad v_4 = (1, 1, -6, -3), \]
    and \( v_5 = (-1, -5, 1, 0) \). Find a basis for the subspace \( W = \text{span} \ S \) of \( \mathbb{R}^4 \). What is \( \dim W \)?

13. Consider the following subset of \( P_3 \):
    \[ S = \{t^3 + t^2 - 2t + 1, t^2 + 1, t^3 - 2t, \]
    \[2t^3 + 3t^2 - 4t + 3\}. \]

Find a basis for the subspace \( W = \text{span} \ S \). What is \( \dim W \)?

14. Let
    \[ S = \begin{bmatrix}
    1 & 0 \\
    0 & 1 \\
    0 & 0
    \end{bmatrix}, \begin{bmatrix}
    0 & 1 \\
    1 & 0 \\
    1 & 1
    \end{bmatrix}, \begin{bmatrix}
    1 & 1 \\
    1 & 0 \\
    1 & 1
    \end{bmatrix} \begin{bmatrix}
    -1 & 1 \\
    1 & -1
    \end{bmatrix}. \]

Find a basis for the subspace \( W = \text{span} \ S \) of \( M_{22} \).

15. Find a basis for \( M_{22} \). What is the dimension of \( M_{22} \)?
    Generalize to \( M_{mn} \).

16. Consider the following subset of the vector space of all real-valued functions
    \[ S = \{\cos^2 t, \sin^2 t, \cos 2t\}. \]

Find a basis for the subspace \( W = \text{span} \ S \). What is \( \dim W \)?

In Exercises 17 and 18, find a basis for the given subspaces of \( \mathbb{R}^3 \) and \( \mathbb{R}^4 \).

17. (a) All vectors of the form \((a, b, c), \) where \( b = a + c \)
    (b) All vectors of the form \((a, b, c), \) where \( b = a \)
    (c) All vectors of the form \((a, b, c), \) where \( 2a + b - c = 0 \)

18. (a) All vectors of the form \((a, b, c), \) where \( a = 0 \)
    (b) All vectors of the form \((a + c, a - b, b + c, -a + b)\)
    (c) All vectors of the form \((a, b, c), \) where \( a - b + 5c = 0 \)

In Exercises 19 and 20, find the dimensions of the given subspaces of \( \mathbb{R}^4 \).

19. (a) All vectors of the form \((a, b, c, d), \) where \( d = a + b \)
    (b) All vectors of the form \((a, b, c, d), \) where \( c = a - b \)
    and \( d = a + b \)

20. (a) All vectors of the form \((a, b, c, d), \) where \( a = b \)
    (b) All vectors of the form \((a + c, -a + b, -b + c, a + b + 2c)\)

21. Find a basis for the subspace of \( P_3 \) consisting of all vectors of the form \( at^2 + bt + c, \) where \( c = 2a - 3b. \)

22. Find a basis for the subspace of \( P_3 \) consisting of all vectors of the form \( at^3 + bt^2 + ct + d, \) where \( a = b \) and \( c = d. \)

23. Find the dimensions of the subspaces of \( \mathbb{R}^2 \) spanned by the vectors in Exercise 1.

24. Find the dimensions of the subspaces of \( \mathbb{R}^3 \) spanned by the vectors in Exercise 2.

25. Find the dimensions of the subspaces of \( \mathbb{R}^4 \) spanned by the vectors in Exercise 3.

26. Find the dimension of the subspace of \( P_3 \) consisting of all vectors of the form \( at^2 + bt + c, \) where \( c = b - 2a. \)

27. Find the dimension of the subspace of \( P_3 \) consisting of all vectors of the form \( at^3 + bt^2 + ct + d, \) where \( b = 3a - 5d \) and \( c = d + 4a. \)

28. Find a basis for \( \mathbb{R}^3 \) that includes the vectors
    (a) \((1, 0, 2)\)
    (b) \((1, 0, 2) \) and \((0, 1, 3)\)

29. Find a basis for \( \mathbb{R}^4 \) that includes the vectors \((1, 0, 1, 0)\)
    and \((0, 1, -1, 0)\).

30. Find all values of \( a \) for which \( \{(a^2, 0, 1), (0, a, 2), \)
    \((1, 0, 1)\} \) is a basis for \( \mathbb{R}^3 \).

31. Find a basis for the subspace \( W \) of \( M_{33} \) consisting of all symmetric matrices.

32. Find a basis for the subspace of \( M_{33} \) consisting of all diagonal matrices.

33. Give an example of a two-dimensional subspace of \( \mathbb{R}^4. \)

34. Give an example of a two-dimensional subspace of \( P_3. \)

In Exercises 35 and 36, find a basis for the given plane.

35. \(2x - 3y + 4z = 0.\)

36. \(x + y - 3z = 0.\)

37. Determine if the vectors
    \[
    \begin{bmatrix}
    1 \\
    1 \\
    1
    \end{bmatrix}, \begin{bmatrix}
    1 \\
    0 \\
    0
    \end{bmatrix}, \begin{bmatrix}
    0 \\
    0 \\
    1
    \end{bmatrix}
    \]
    are a basis for \( \mathbb{R}^3 \).
Theoretical Exercises

T.1. Suppose that in the nonzero vector space $V$, the largest number of vectors in a linearly independent set is $m$. Show that any set of $m$ linearly independent vectors in $V$ is a basis for $V$.

T.2. Show that if $V$ is a finite-dimensional vector space, then every nonzero subspace $W$ of $V$ has a finite basis and $\dim W \leq \dim V$.

T.3. Show that if $\dim V = n$, then any $n + 1$ vectors in $V$ are linearly dependent.

T.4. Show that if $\dim V = n$, then no set of $n + 1$ vectors in $V$ can span $V$.


T.7. Show that if $W$ is a subspace of a finite-dimensional vector space $V$ and $\dim W = \dim V$, then $W = V$.

T.8. Show that the subspaces of $R^3$ are $\{0\}$, $R^3$, all lines through the origin, and all planes through the origin.

T.9. Show that if $\{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space $V$ and $c \neq 0$, then $\{cv_1, v_2, \ldots, v_n\}$ is also a basis for $V$.

T.10. Let $S = \{v_1, v_2, v_3\}$ be a basis for vector space $V$. Then show that $T = \{w_1, w_2, w_3\}$, where

\[
\begin{align*}
    w_1 &= v_1 + v_2 + v_3, \\
    w_2 &= v_2 + v_3, \\
    w_3 &= v_3,
\end{align*}
\]

is also a basis for $V$.

T.11. Let

\[ S = \{v_1, v_2, \ldots, v_n\} \]

be a set of nonzero vectors in a vector space $V$ such that every vector in $V$ can be written in one and only one way as a linear combination of the vectors in $S$. Show that $S$ is a basis for $V$.

T.12. Suppose that $\{v_1, v_2, \ldots, v_n\}$ is a basis for $R^n$. Show that if $A$ is an $n \times n$ nonsingular matrix, then $\{Av_1, Av_2, \ldots, Av_n\}$ is also a basis for $R^n$. (Hint: See Exercise T.10 in Section 6.3.)

T.13. Suppose that $\{v_1, v_2, \ldots, v_n\}$ is a linearly independent set of vectors in $R^n$ and let $A$ be a singular matrix. Prove or disprove that $\{Av_1, Av_2, \ldots, Av_n\}$ is linearly independent.

T.14. Show that the vector space $P$ of all polynomials is not finite-dimensional. (Hint: Suppose that $\{p_1(t), p_2(t), \ldots, p_k(t)\}$ is a finite basis for $P$. Let $d = \deg p_i(t)$. Establish a contradiction.)

T.15. Show that the set of vectors $\{t^n, t^{n-1}, \ldots, t, 1\}$ in $P_n$ is linearly independent.

T.16. Show that if the sum of the vectors $v_1, v_2, \ldots, v_n$ from $B^n$ is 0, then these vectors cannot form a basis for $B^n$.

T.17. Let $S = \{v_1, v_2, v_3\}$ be a set of vectors in $B^3$.

(a) Find linearly independent vectors $v_1, v_2, v_3$ such that $v_1 + v_2 + v_3 \neq 0$.

(b) Find linearly dependent vectors $v_1, v_2, v_3$ such that $v_1 + v_2 + v_3 \neq 0$. 

38. Determine if the vectors

\[
\begin{bmatrix}
    1 \\
    0 \\
    0
\end{bmatrix}, 
\begin{bmatrix}
    1 \\
    0 \\
    1
\end{bmatrix}, 
\begin{bmatrix}
    0 \\
    1 \\
    0
\end{bmatrix}
\]

are a basis for $B^3$.

39. Determine if the vectors

\[
\begin{bmatrix}
    1 \\
    0 \\
    0
\end{bmatrix}, 
\begin{bmatrix}
    0 \\
    0 \\
    1
\end{bmatrix}, 
\begin{bmatrix}
    0 \\
    1 \\
    1
\end{bmatrix}
\]

are a basis for $B^4$.

40. Determine if the vectors

\[
\begin{bmatrix}
    1 \\
    1 \\
    0
\end{bmatrix}, 
\begin{bmatrix}
    0 \\
    1 \\
    0
\end{bmatrix}, 
\begin{bmatrix}
    0 \\
    0 \\
    1
\end{bmatrix}
\]

are a basis for $B^4$. 

MATLAB Exercises

In order to use MATLAB in this section, you should have read Section 12.7. In the following exercises we relate the theory developed in the section to computational procedures in MATLAB that aid in analyzing the situation.

To determine if a set \( S = \{v_1, v_2, \ldots, v_k\} \) is a basis for a vector space \( V \), the definition requires that we show \( \text{span} \ S = V \) and \( S \) is linearly independent. However, if we know that \( \dim V = k \), then Theorem 6.9 implies that we need only show that either \( \text{span} \ S = V \) or \( S \) is linearly independent. The linear independence, in this special case, is easily analyzed using MATLAB's \texttt{rref} command. Construct the homogeneous system \( Ax = 0 \) associated with the linear independence/dependence question. Then \( S \) is linearly independent if and only if

\[
\text{rref}(A) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

In Exercises ML.1 through ML.6, if this special case can be applied, do so; otherwise, determine if \( S \) is a basis for \( V \) in the conventional manner.

ML.1. \( S = \{(1, 2, 1), (2, 1, 1), (2, 2, 1)\} \) in \( V = \mathbb{R}^3 \)

ML.2. \( S = \{2t - 2, t^2 - 3t + 1, 2t^2 - 8t + 4\} \) in \( V = P_2 \)

ML.3. \( S = \{(1, 1, 0, 0), (2, 1, 1, -1), (0, 0, 1, 1), (1, 2, 1, 2)\} \) in \( V = \mathbb{R}^4 \)

ML.4. \( S = \{(1, 2, 1, 0), (2, 1, 3, 1), (2, -2, 4, 2)\} \) in \( V = \text{span} \ S \)

ML.5. \( S = \{(1, 2, 1, 0), (2, 1, 3, 1), (2, 2, 1, 2)\} \) in \( V = \text{span} \ S \)

ML.6. \( V = \) the subspace of \( \mathbb{R}^3 \) of all vectors of the form \((a, b, c)\), where \( b = 2a - c \) and \( S = \{(0, 1, -1), (1, 1, 1)\} \).

In Exercises ML.7 through ML.9, use MATLAB's \texttt{rref} command to determine a subset of \( S \) that is a basis for \( \text{span} \ S \). See Example 5.

ML.7. \( S = \{(1, 1, 0, 0), (-2, -2, 0, 0), (1, 0, 2, 1), (2, 1, 2, 1), (0, 1, 1, 1)\} \).

What is \( \dim \text{span} \ S \)? Does \( \text{span} \ S = \mathbb{R}^4 \)?

ML.8. \( S = \begin{bmatrix} 1 & 2 & 1 & 0 \\
1 & 2 & 1 & 0 \\
2 & 4 & 1 & 0 \end{bmatrix} \).

What is \( \dim \text{span} \ S \)? Does \( \text{span} \ S = M_{22} \)?

ML.9. \( S = \{t - 2, 2t - 1, 4t - 2, t^2 - t + 1, t^2 + 2t + 1\} \).

What is \( \dim \text{span} \ S \)? Does \( \text{span} \ S = P_2 \)?

An interpretation of Theorem 6.8 is that any linearly independent subset \( S \) of vector space \( V \) can be extended to a basis for \( V \). Following the ideas in Example 9, use MATLAB's \texttt{rref} command to extend \( S \) to a basis for \( V \) in Exercises ML.10 through ML.12.

ML.10. \( S = \{(1, 1, 0, 0), (1, 0, 1, 0)\}, \ V = \mathbb{R}^4 \)

ML.11. \( S = \{t^3 - t + 1, t^3 + 2\}, \ V = P_3 \)

ML.12. \( S = \{(0, 3, 0, 2, -1)\}, \ V = \) the subspace of \( \mathbb{R}^5 \) consisting of all vectors of the form \((a, b, c, d, e)\), where \( c = a, b = 2d + e \)

### 6.5 Homogeneous Systems

Homogeneous systems play a central role in linear algebra. This will be seen in Chapter 8, where the foundations of the subject are all integrated to solve one of the major problems occurring in a wide variety of applications. In this section we deal with several problems involving homogeneous systems that will be fundamental in Chapter 8. Here we are able to focus our attention on these problems without being distracted by the additional material in Chapter 8.

Consider the homogeneous system

\[
Ax = 0,
\]

where \( A \) is an \( m \times n \) matrix. As we have already observed in Example 9 of Section 6.2, the set of all solutions to this homogeneous system is a subspace of \( \mathbb{R}^n \). An extremely important problem, which will occur repeatedly in Chapter 8, is that of finding a basis for this solution space. To find such a basis, we use the method of Gauss-Jordan reduction presented in Section 1.6. Thus we transform the augmented matrix \( [A : 0] \) of the system to a matrix \( [B : 0] \) in reduced row echelon form, where \( B \) has \( r \) nonzero rows, \( 1 \leq r \leq m \). Without
6.5 Exercises

1. Let
\[
A = \begin{bmatrix}
  2 & -1 & -2 \\
  -4 & 2 & 4 \\
  -8 & 4 & 8 \\
\end{bmatrix}.
\]
(a) Find the set of all solutions to \( Ax = 0 \).
(b) Express each solution as a linear combination of two vectors in \( \mathbb{R}^3 \).
(c) Sketch these vectors in a three-dimensional coordinate system to show that the solution space is a plane through the origin.

2. Let
\[
A = \begin{bmatrix}
  1 & 1 & -2 \\
  -2 & -2 & 4 \\
  -1 & -1 & 2 \\
\end{bmatrix}.
\]
(a) Find the set of all solutions to \( Ax = 0 \).
(b) Express each solution as a linear combination of two vectors in \( \mathbb{R}^3 \).
(c) Sketch these vectors in a three-dimensional coordinate system to show that the solution space is a plane through the origin.

In Exercises 3 through 10, find a basis for and the dimension of the solution space of the given homogeneous system.

3. \( x_1 + x_2 + x_3 + x_4 = 0 \)
\( 2x_1 + x_2 - x_3 + x_4 = 0 \)

4. \[
\begin{bmatrix}
  1 & 1 & -2 & 1 \\
  3 & -3 & 2 & 2 \\
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0 \\
\end{bmatrix}
\]

5. \( x_1 + 2x_2 - x_3 + 3x_4 = 0 \)
\( 2x_1 + x_2 - x_3 + 2x_4 = 0 \)
\( x_1 + 3x_3 + 3x_4 = 0 \)

6. \( x_1 - x_2 + 2x_3 + 3x_4 + 4x_5 = 0 \)
\( -x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 0 \)
\( x_1 - x_2 + 3x_3 + 5x_4 + 6x_5 = 0 \)
\( 3x_1 - 4x_2 + x_3 + 2x_4 + 3x_5 = 0 \)

7. \[
\begin{bmatrix}
  1 & 2 & 1 & 1 & 2 & 1 \\
  2 & 4 & 3 & 3 & 3 & 3 \\
  0 & 0 & 1 & -1 & -1 & 2 \\
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0 \\
  0 \\
\end{bmatrix}
\]

8. \[
\begin{bmatrix}
  1 & 0 & 2 \\
  2 & 1 & 3 \\
  3 & 1 & 2 \\
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0 \\
\end{bmatrix}
\]

9. \[
\begin{bmatrix}
  1 & 2 & 2 & -1 & 1 & 1 \\
  0 & 2 & 2 & -2 & -1 & 2 \\
  2 & 6 & 2 & -4 & 1 & 1 \\
  1 & 4 & 0 & -3 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0 \\
  0 \\
  0 \\
\end{bmatrix}
\]

10. \[
\begin{bmatrix}
  1 & 2 & -3 & 2 & 1 & -3 \\
  1 & 2 & -4 & 3 & 3 & 4 \\
  0 & 0 & -1 & 5 & 1 & 9 \\
  1 & 2 & -3 & -2 & 0 & 7 \\
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
  x_6 \\
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0 \\
  0 \\
  0 \\
\end{bmatrix}
\]

In Exercises 11 and 12, find a basis for the null space of the given matrix \( A \).

11. \( A = \begin{bmatrix}
  1 & 2 & 3 & -1 \\
  2 & 3 & 2 & 0 \\
  3 & 4 & 1 & 1 \\
  1 & 1 & -1 & 1 \\
\end{bmatrix} \)

12. \( A = \begin{bmatrix}
  1 & -1 & 2 & 1 & 0 \\
  2 & 0 & 1 & -1 & 3 \\
  5 & -1 & 3 & 0 & 3 \\
  4 & -2 & 5 & 1 & 3 \\
  1 & 3 & -4 & -5 & 6 \\
\end{bmatrix} \)

In Exercises 13 through 16, find a basis for the solution space of the homogeneous system \( (\lambda I_n - A)x = 0 \) for the given scalar \( \lambda \) and given matrix \( A \).

13. \( \lambda = 1 \), \( A = \begin{bmatrix}
  3 & 2 \\
  1 & 2 \\
\end{bmatrix} \)

14. \( \lambda = -3 \), \( A = \begin{bmatrix}
  -4 & -3 \\
  2 & 3 \\
\end{bmatrix} \)

15. \( \lambda = 1 \), \( A = \begin{bmatrix}
  0 & 0 & 1 \\
  1 & 0 & -3 \\
  0 & 1 & 3 \\
\end{bmatrix} \)

16. \( \lambda = 3 \), \( A = \begin{bmatrix}
  1 & 1 & -2 \\
  -1 & 2 & 1 \\
  0 & 1 & -1 \\
\end{bmatrix} \)

In Exercises 17 through 20, find all real numbers \( \lambda \) such that the homogeneous system \( (\lambda I_n - A)x = 0 \) has a nontrivial solution.

17. \( A = \begin{bmatrix}
  2 & 3 \\
  2 & -3 \\
\end{bmatrix} \)

18. \( A = \begin{bmatrix}
  3 & 0 \\
  2 & -2 \\
\end{bmatrix} \)

19. \( A = \begin{bmatrix}
  0 & 0 & 0 \\
  0 & 1 & -1 \\
  1 & 0 & 0 \\
\end{bmatrix} \)

20. \( A = \begin{bmatrix}
  -2 & 0 & 0 \\
  0 & -2 & -3 \\
  0 & 4 & 5 \\
\end{bmatrix} \)

In Exercises 21 and 22, solve the given linear system and write the solution \( x \) as \( x = x_p + x_h \), where \( x_p \) is a particular solution to the given system and \( x_h \) is a solution to the associated homogeneous system.

21. \( x + 2y - z - w = 3 \)
\( x + y + 3z + 2w = -2 \)
\( 2x - y + 4z + 3w = 1 \)
\( 2x - 2y + 8z + 6w = -4 \)

22. \( x - y + 2z + 2w = 1 \)
\( -x + 2y + 3z + 2w = 0 \)
\( 2x + 2y + z = 4 \)
In Exercises 23 through 26, find a basis for and the dimension of the solution space of the given bit homogeneous system.

23. \( x_1 + x_2 + x_4 = 0 \)
\( x_1 + x_3 + x_4 = 0 \)
\( x_1 + x_2 + x_3 = 0 \)

\[
\begin{bmatrix}
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]

24. \( \begin{bmatrix}
1 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]

Theoretical Exercises

T.1. Let \( S = \{x_1, x_2, \ldots, x_k\} \) be a set of solutions to a homogeneous system \( Ax = 0 \). Show that every vector in span \( S \) is a solution to \( Ax = 0 \).

T.2. Show that if the \( n \times n \) coefficient matrix \( A \) of the homogeneous system \( Ax = 0 \) has a row or column of zeros, then \( Ax = 0 \) has a nontrivial solution.

T.3. (a) Show that the zero matrix is the only \( 3 \times 3 \) matrix whose null space has dimension 3.

(b) Let \( A \) be a nonzero \( 3 \times 3 \) matrix and suppose that \( Ax = 0 \) has a nontrivial solution. Show that the dimension of the null space of \( A \) is either 1 or 2.

T.4. Matrices \( A \) and \( B \) are \( m \times n \) and their reduced row echelon forms are the same. What is the relationship between the null space of \( A \) and the null space of \( B \)?

MATLAB Exercises

In Exercises ML.1 through ML.3, use MATLAB's \texttt{rref} command to aid in finding a basis for the null space of \( A \). You may also use routine \texttt{homsoin}. For directions, use \texttt{help}.

ML.1. \( A = \begin{bmatrix}
1 & 1 & 2 & 2 & 1 \\
2 & 0 & 4 & 2 & 4 \\
1 & 1 & 2 & 2 & 1 \\
2 & 2 & 2 & 1 \\
1 & 2 & 1 \\
3 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \]

ML.2. \( A = \begin{bmatrix}
1 & 4 & 7 & 0 \\
2 & 5 & 8 & -1 \\
3 & 6 & 9 & -2 \\
\end{bmatrix} \]

ML.3. \( A = \begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 2 & 1 \\
\end{bmatrix} \]

ML.4. For the matrix
\[
A = \begin{bmatrix}
1 & 2 \\
2 & 1 \\
\end{bmatrix}
\]
and \( \lambda = 3 \), the homogeneous system \((\lambda I_3 - A)x = 0\) has a nontrivial solution. Find such a solution using MATLAB commands.

ML.5. For the matrix
\[
A = \begin{bmatrix}
1 & 2 & 3 \\
3 & 2 & 1 \\
2 & 1 & 3 \\
\end{bmatrix}
\]
and \( \lambda = 6 \), the homogeneous linear system \((\lambda I_3 - A)x = 0\) has a nontrivial solution. Find such a solution using MATLAB commands.

6.6 THE RANK OF A MATRIX AND APPLICATIONS

In this section we obtain another effective method for finding a basis for a vector space \( V \) spanned by a given set of vectors \( S = \{v_1, v_2, \ldots, v_n\} \). In the proof of Theorem 6.6 we developed a technique for choosing a basis for \( V \) that is a subset of \( S \). The method to be developed in this section produces a basis for \( V \) that is not guaranteed to be a subset of \( S \). We shall also attach a unique number to a matrix \( A \) that we later show gives us information about
EXAMPLE 9
Consider the linear system
\[
\begin{bmatrix}
2 & 1 & 3 \\
1 & -2 & 2 \\
0 & 1 & 3 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
= \begin{bmatrix}
1 \\
2 \\
3 \\
\end{bmatrix}.
\]
Since \( \text{rank } A = \text{rank } [A \mid b] = 3 \), the linear system has a solution.

EXAMPLE 10
The linear system
\[
\begin{bmatrix}
1 & 2 & 3 \\
1 & -3 & 4 \\
2 & -1 & 7 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
= \begin{bmatrix}
4 \\
5 \\
6 \\
\end{bmatrix}
\]
has no solution because \( \text{rank } A = 2 \) and \( \text{rank } [A \mid b] = 3 \) (verify).

We now extend our list of nonsingular equivalences.

**List of Nonsingular Equivalences**
The following statements are equivalent for an \( n \times n \) matrix \( A \).

1. \( A \) is nonsingular.
2. \( x = 0 \) is the only solution to \( Ax = 0 \).
3. \( A \) is row equivalent to \( I_n \).
4. The linear system \( Ax = b \) has a unique solution for every \( n \times 1 \) matrix \( b \).
5. \( \text{det}(A) \neq 0 \).
6. \( A \) has rank \( n \).
7. \( A \) has nullity 0.
8. The rows of \( A \) form a linearly independent set of \( n \) vectors in \( \mathbb{R}^n \).
9. The columns of \( A \) form a linearly independent set of \( n \) vectors in \( \mathbb{R}^n \).

**Key Terms**

<table>
<thead>
<tr>
<th>Row space</th>
<th>Column rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>Column space</td>
<td>Rank</td>
</tr>
<tr>
<td>Row rank</td>
<td>Nonhomogeneous system</td>
</tr>
</tbody>
</table>

**6.6 Exercises**

1. Let \( S = \{v_1, v_2, v_3, v_4, v_5\} \), where
   \[
   v_1 = (1, 2, 3), \quad v_2 = (2, 1, 4),
   \]
   and \( v_3 = (1, 1, 1) \). Find a basis for the subspace \( V = \text{span } S \) of \( \mathbb{R}^3 \).

2. Let \( S = \{v_1, v_2, v_3, v_4, v_5\} \), where
   \[
   v_1 = (1, 1, 2, 1), \quad v_2 = (1, 0, -3, 1),
   v_3 = (0, 1, 1, 2), \quad v_4 = (0, 0, 1, 1),
   v_5 = (1, 0, 0, 1),
   \]
   and \( v_3 = (1, 0, 0, 1) \). Find a basis for the subspace \( V = \text{span } S \) of \( \mathbb{R}^4 \).

3. Let \( S = \{v_1, v_2, v_3, v_4, v_5\} \), where
   \[
   v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 3 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 3 \\ 3 \\ 5 \\ 5 \end{bmatrix}.
   \]
   Find a basis for the subspace \( V = \text{span } S \) of \( \mathbb{R}^4 \).
4. Let $S = \{v_1, v_2, v_3, v_4, v_5\}$, where

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix},$$

$$v_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix}, \quad \text{and} \quad v_5 = \begin{bmatrix} 5 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

Find a basis for the subspace $V = \text{span} S$ of $\mathbb{R}^4$.

In Exercises 5 and 6, find a basis for the row space of $A$

(a) consisting of vectors that are not row vectors of $A$;

(b) consisting of vectors that are row vectors of $A$.

5. $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 9 & -1 \\ -3 & 8 & 3 \\ -2 & 3 & 2 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 5 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$

In Exercises 7 and 8, find a basis for the column space of $A$

(a) consisting of vectors that are not column vectors of $A$;

(b) consisting of vectors that are column vectors of $A$.

7. $A = \begin{bmatrix} 1 & -2 & 7 & 0 \\ 1 & -1 & 4 & 0 \\ 3 & 2 & -3 & 5 \\ 2 & 1 & -1 & 3 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & -2 & 2 & 3 & 7 & 1 \\ -2 & 2 & 4 & 8 & 0 \\ -3 & 3 & 2 & 8 & 4 \\ 4 & -2 & 1 & -5 & -7 \end{bmatrix}$

In Exercises 9 and 10, compute a basis for the row space of $A$, the column space of $A$, the row space of $A^T$, and the column space of $A^T$. Write a short description giving the relationships among these bases.

9. $A = \begin{bmatrix} 1 & -2 & 5 \\ 2 & 3 & 2 \\ 0 & -7 & 8 \end{bmatrix}$

10. $A = \begin{bmatrix} 2 & -3 & -7 & 11 \\ 3 & -1 & -7 & 13 \\ 1 & 2 & 0 & 2 \end{bmatrix}$

In Exercises 11 and 12, compute the row and column ranks of $A$, verifying Theorem 6.11.

11. $A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 1 & -5 & -2 & 1 \\ 7 & 8 & -1 & 2 & 5 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 3 & 2 & 0 & 0 & 1 \\ 2 & 1 & 5 & 1 & 2 & 0 \\ 3 & 2 & 5 & 1 & -2 & 1 \\ 5 & 8 & 9 & 1 & -2 & 2 \\ 9 & 4 & 2 & 0 & 2 \end{bmatrix}$

In Exercises 13 through 17, compute the rank and nullity of $A$ and verify Theorem 6.12.

13. $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & -4 & -5 \\ 7 & 8 & -5 & -1 \\ 10 & 14 & -2 & 8 \end{bmatrix}$

14. $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & -4 & -5 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

15. $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

16. $A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & -1 & 3 \\ 7 & -8 & 3 \\ 5 & -7 & 0 \end{bmatrix}$

18. If $A$ is a $3 \times 4$ matrix, what is the largest possible value for rank $A$?

19. If $A$ is a $4 \times 6$ matrix, show that the columns of $A$ are linearly dependent.

20. If $A$ is a $5 \times 3$ matrix, show that the rows of $A$ are linearly dependent.

In Exercises 21 and 22, let $A$ be a $7 \times 3$ matrix whose rank is 3.

21. Are the rows of $A$ linearly dependent or linearly independent? Justify your answer.

22. Are the columns of $A$ linearly dependent or linearly independent? Justify your answer.

In Exercises 23 through 25, use Theorem 6.13 to determine whether each matrix is singular or nonsingular.

23. $\begin{bmatrix} 1 & 2 & -3 \\ -1 & 2 & 3 \\ 0 & 8 & 0 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 2 & -3 \\ -1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$

25. $\begin{bmatrix} 1 & 1 & 4 & -1 \\ 1 & 2 & 3 & 2 \\ -1 & 3 & 2 & 1 \\ -2 & 6 & 12 & -4 \end{bmatrix}$

In Exercises 26 and 27, use Corollary 6.3 to determine whether the linear system $Ax = b$ has a unique solution for every $3 \times 1$ matrix $b$. 

26. \( A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 8 & -7 \\ 3 & -2 & 1 \end{bmatrix} \)

27. \( A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 3 \\ 1 & -2 & 1 \end{bmatrix} \)

Use Corollary 6.4 to do Exercises 28 and 29.

28. Is \[ S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\} \]
a linearly independent set of vectors in \( R^3 \)?

29. Is \[ S = \left\{ \begin{bmatrix} 4 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \\ 3 \end{bmatrix} \right\} \]
a linearly independent set of vectors in \( R^4 \)?

In Exercises 30 through 32, find which homogeneous systems have a nontrivial solution for the given matrix \( A \) by using Corollary 6.5.

30. \( A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & 3 & -1 & 2 \\ 1 & 1 & 1 & 3 \\ 1 & 2 & 1 & 1 \end{bmatrix} \)

31. \( A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix} \)

32. \( A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 5 & -4 & 3 \end{bmatrix} \)

In Exercises 33 through 36, determine which of the linear systems have a solution by using Theorem 6.14.

33. \[ \begin{bmatrix} 1 & 2 & 5 & -2 \\ 2 & 3 & -2 & 4 \\ 5 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

34. \[ \begin{bmatrix} 1 & 2 & 5 & -2 \\ 2 & 3 & -2 & 4 \\ 5 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -13 \\ 3 \end{bmatrix} \]

35. \[ \begin{bmatrix} 1 & -2 & -3 & 4 \\ 4 & -1 & -5 & 6 \\ 2 & 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \]

36. \[ \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 5 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix} \]

In Exercises 37 through 40, compute the rank of the given bit matrix.

37. \[ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \]

38. \[ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \]

39. \[ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \]

40. \[ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \]

Theoretical Exercises

T1. Prove Corollary 6.2.

T2. Prove Corollary 6.3.


T4. Let \( A \) be an \( n \times n \) matrix. Show that the homogeneous system \( Ax = 0 \) has a nontrivial solution if and only if the columns of \( A \) are linearly dependent.

T5. Let \( A \) be an \( n \times n \) matrix. Show that \( \text{rank} \ A = n \) if and only if the columns of \( A \) are linearly independent.

T6. Let \( A \) be an \( n \times n \) matrix. Show that the rows of \( A \) are linearly independent if and only if the columns of \( A \) span \( R^n \).

T7. Let \( A \) be an \( m \times n \) matrix. Show that the linear system \( Ax = b \) has a solution for every \( m \times 1 \) matrix \( b \) if and only if \( \text{rank} \ A = m \).

T8. Let \( A \) be an \( m \times n \) matrix. Show that the columns of \( A \) are linearly independent if and only if the homogeneous system \( Ax = 0 \) has only the trivial solution.

T9. Let \( A \) be an \( m \times n \) matrix. Show that the linear system \( Ax = b \) has at most one solution for every \( m \times 1 \) matrix \( b \) if and only if the associated homogeneous system \( Ax = 0 \) has only the trivial solution.

T10. Let \( A \) be an \( m \times n \) matrix with \( m \neq n \). Show that either the rows or the columns of \( A \) are linearly dependent.

T11. Suppose that the linear system \( Ax = b \), where \( A \) is \( m \times n \), is consistent (has a solution). Show that the solution is unique if and only if \( \text{rank} \ A = n \).

T12. Show that if \( A \) is an \( m \times n \) matrix such that \( AA^T \) is nonsingular, then \( \text{rank} \ A = m \).
MATLAB Exercises

Given a matrix $A$, the nonzero rows of $\text{rref}(A)$ form a basis for the row space of $A$ and the nonzero rows of $\text{rref}(A')$ transformed to columns give a basis for the column space of $A$.

ML.1. Solve Exercises 1 through 4 using MATLAB.

To find a basis for the row space of $A$ that consists of rows of $A$, we compute $\text{rref}(A')$. The leading 1s point to the original rows of $A$ that give us a basis for the row space. See Example 3.

ML.2. Determine two bases for the row space of $A$ that have no vectors in common.

(a) $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 0 \\ 4 & 11 & 2 \\ 6 & 9 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 \\ 4 & 5 & 6 & 2 \\ 3 & 3 & 4 & 1 \end{bmatrix}$

ML.3. Repeat Exercise ML.2 for column spaces.

To compute the rank of a matrix $A$ in MATLAB, use the command $\text{rank}(A)$.

ML.4. Compute the rank and nullity of each of the following matrices.

(a) $A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 0 & 0 & 2 \\ 1 & -1 & -1 & -2 & 1 \\ 3 & 0 & -1 & -2 & 3 \end{bmatrix}$

ML.5. Using only the rank command, determine whether the following linear systems are consistent.

(a) $A = \begin{bmatrix} 1 & 2 & 4 & -1 \\ 0 & 1 & 2 & 0 \\ 3 & 1 & 1 & -2 \end{bmatrix}$ $x = \begin{bmatrix} 21 \\ 8 \\ 16 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$ $x = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$ $x = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$

6.7 COORDINATES AND CHANGE OF BASIS

coordinates

If $V$ is an $n$-dimensional vector space, we know that $V$ has a basis $S$ with vectors in it; so far we have not paid much attention to the order of the vectors in $S$. However, in the discussion of this section we speak of an ordered basis $S = \{v_1, v_2, \ldots, v_n\}$ for $V$; thus $S_1 = \{v_2, v_1, \ldots, v_n\}$ is a different ordered basis for $V$.

If $S = \{v_1, v_2, \ldots, v_n\}$ is an ordered basis for the $n$-dimensional vector space $V$, then every vector $v$ in $V$ can be uniquely expressed in the form

$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$,

where $c_1, c_2, \ldots, c_n$ are real numbers. We shall refer to

$[v]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

as the coordinate vector of $v$ with respect to the ordered basis $S$. The entries of $[v]_S$ are called the coordinates of $v$ with respect to $S$.

Observe that the coordinate vector $[v]_S$ depends on the order in which the vectors in $S$ are listed; a change in the order of this listing may change the coordinates of $v$ with respect to $S$. All bases considered in this section are assumed to be ordered bases.
8.1 Exercises

1. Let $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$.

(a) Verify that $\lambda_1 = 1$ is an eigenvalue of $A$ and 
$x_1 = \begin{bmatrix} r \\ 2r \end{bmatrix}, r \neq 0$, is an associated eigenvector.

(b) Verify that $\lambda_2 = 4$ is an eigenvalue of $A$ and 
$x_2 = \begin{bmatrix} r \\ -r \end{bmatrix}, r \neq 0$, is an associated eigenvector.

2. Let $A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix}$.

(a) Verify that $\lambda_1 = -1$ is an eigenvalue of $A$ and 
$x_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is an associated eigenvector.

(b) Verify that $\lambda_2 = 2$ is an eigenvalue of $A$ and 
$x_2 = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$ is an associated eigenvector.

(c) Verify that $\lambda_3 = 4$ is an eigenvalue of $A$ and 
$x_3 = \begin{bmatrix} 8 \\ 5 \\ 2 \end{bmatrix}$ is an associated eigenvector.

In Exercises 3 through 7, find the characteristic polynomial of each matrix.

3. $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -1 & 3 & 2 \end{bmatrix}$

4. $\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

5. $\begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix}$

In Exercises 8 through 15, find the characteristic polynomial, eigenvalues, and eigenvectors of each matrix.

8. $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 3 & 2 & -2 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

11. $\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$

12. $\begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$

13. $\begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & -2 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ 0 & 4 & 3 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 3 \\ 0 & 0 & 3 \end{bmatrix}$

16. Find the characteristic polynomial, the eigenvalues and associated eigenvectors of each of the following matrices.

(a) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} -2 & -4 & -8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 2 - i & 2i & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix}$

17. Find all the eigenvalues and associated eigenvectors of each of the following matrices.

(a) $\begin{bmatrix} -1 & -1 + i \\ 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} i & 1 & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0 & -9 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

In Exercises 18 and 19, find bases for the eigenspaces (see Exercise T.1) associated with each eigenvalue.

18. $\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

19. $\begin{bmatrix} 2 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

In Exercises 20 through 23, find a basis for the eigenspace (see Exercise T.1) associated with $\lambda$.

20. $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$, $\lambda = 1$

21. $\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$, $\lambda = 2$

22. $\begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & -2 \\ 2 & 0 & 5 \end{bmatrix}$, $\lambda = 3$

23. $\begin{bmatrix} 4 & 2 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $\lambda = 2$
24. Let \( A = \begin{bmatrix} 0 & -4 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \).

(a) Find a basis for the eigenspace associated with the eigenvalue \( \lambda_1 = 2i \).

(b) Find a basis for the eigenspace associated with the eigenvalue \( \lambda_2 = -2i \).

25. Let \( A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 3 \\ 0 & 0 & -3 & 0 \end{bmatrix} \).

(a) Find a basis for the eigenspace associated with the eigenvalue \( \lambda_1 = 3 \).

(b) Find a basis for the eigenspace associated with the eigenvalue \( \lambda_2 = 3i \).

26. Let \( A \) be the matrix of Exercise 1. Find the eigenvalues and eigenvectors of \( A^2 \) and verify Exercise T.5.

27. Consider a living organism that can live to a maximum age of 2 years and whose Leslie matrix is

\[ A = \begin{bmatrix} 0 & 0 & 8 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \]

Find a stable age distribution.

28. Consider a living organism that can live to a maximum age of 2 years and whose Leslie matrix is

\[ A = \begin{bmatrix} 0 & 4 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \]

Show that a stable age distribution exists and find one.

---

**Theoretical Exercises**

T.1. Let \( \lambda_j \) be a particular eigenvalue of the \( n \times n \) matrix \( A \). Show that the subset \( S \) of \( \mathbb{R}^n \) consisting of the zero vector and all eigenvectors of \( A \) associated with \( \lambda_j \) is a subspace of \( \mathbb{R}^n \), called the eigenspace associated with the eigenvalue \( \lambda_j \).

T.2. In Exercise T.1 why do we have to include the zero vector in the subset \( S \)?

T.3. Show that if \( A \) is an upper (lower) triangular matrix or a diagonal matrix, then the eigenvalues of \( A \) are the elements on the main diagonal of \( A \).

T.4. Show that \( A \) and \( A^T \) have the same eigenvalues. What, if anything, can we say about the associated eigenvectors of \( A \) and \( A^T \)?

T.5. If \( \lambda \) is an eigenvalue of \( A \) with associated eigenvector \( x \), show that \( \lambda^k \) is an eigenvalue of \( A^k = A \cdot A \cdot \cdots A \) (\( k \) factors) with associated eigenvector \( x \), where \( k \) is a positive integer.

T.6. An \( n \times n \) matrix \( A \) is called nilpotent if \( A^k = 0 \) for some positive integer \( k \). Show that if \( A \) is nilpotent, then the only eigenvalue of \( A \) is 0. (Hint: Use Exercise T.5.)

T.7. Let \( A \) be an \( n \times n \) matrix.

(a) Show that \( \text{det}(A) \) is the product of all the roots of the characteristic polynomial of \( A \).

(b) Show that \( A \) is singular if and only if 0 is an eigenvalue of \( A \).

T.8. Let \( \lambda \) be an eigenvalue of the nonsingular matrix \( A \) with associated eigenvector \( x \). Show that \( 1/\lambda \) is an eigenvalue of \( A^{-1} \) with associated eigenvector \( x \).

T.9. Let \( A \) be any \( n \times n \) real matrix.

(a) Show that the coefficient of \( \lambda^{n-1} \) in the characteristic polynomial of \( A \) is given by \(-\text{Tr}(A)\), where \( \text{Tr}(A) \) denotes the trace of \( A \) (see Supplementary Exercise T.1 in Chapter 1).

(b) Show that \( \text{Tr}(A) \) is the sum of the eigenvalues of \( A \).

(c) Show that the constant term of the characteristic polynomial of \( A \) is \( \pm \) times the product of the eigenvalues of \( A \).

(d) Show that \( \text{det}(A) \) is the product of the eigenvalues of \( A \).

T.10. Let \( A \) be an \( n \times n \) matrix with eigenvalues \( \lambda_1 \) and \( \lambda_2 \), where \( \lambda_1 \neq \lambda_2 \). Let \( S_1 \) and \( S_2 \) be the eigenspaces associated with \( \lambda_1 \) and \( \lambda_2 \), respectively. Explain why the zero vector is the only vector that is in both \( S_1 \) and \( S_2 \).

T.11. Let \( \lambda \) be an eigenvalue of \( A \) with associated eigenvector \( x \). Show that \( \lambda + r \) is an eigenvalue of \( A + rI_n \) with associated eigenvector \( x \). Thus, adding a scalar multiple of the identity matrix to \( A \) merely shifts the eigenvalues by the scalar multiple.

T.12. Let \( A \) be a square matrix.

(a) Suppose that the homogeneous system \( Ax = 0 \) has a nontrivial solution \( x = u \). Show that \( u \) is an eigenvector of \( A \).

(b) Suppose that \( 0 \) is an eigenvalue of \( A \) and \( v \) is an associated eigenvector. Show that the homogeneous system \( Ax = 0 \) has a nontrivial solution.

T.13. Let \( A \) and \( B \) be \( n \times n \) matrices such that \( Ax = \lambda x \) and \( Bx = \mu x \). Show that:

(a) \((A + B)x = (\lambda + \mu)x\)

(b) \((AB)x = (\lambda \mu)x\)
MATLAB Exercises

MATLAB has a pair of commands that can be used to find the characteristic polynomial and eigenvalues of a matrix.

Command **poly(A)** gives the coefficients of the characteristic polynomial of matrix A, starting with the highest-degree term. If we set v = poly(A) and then use command roots(v), we obtain the roots of the characteristic polynomial of A. This process can also find complex eigenvalues, which are discussed in Appendix A.2.

Once we have an eigenvalue \( \lambda \) of A, we can use `rref` or `homsolv` to find a corresponding eigenvector from the linear system \((\lambda I - A)x = 0\).

**ML.1.** Find the characteristic polynomial of each of the following matrices using MATLAB.

(a) \( A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \)

(b) \( A = \begin{bmatrix} 2 & 4 & 0 \\ 1 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \)

(c) \( A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \)

**ML.2.** Use the **poly** and **roots** commands in MATLAB to find the eigenvalues of the following matrices:

(a) \( A = \begin{bmatrix} 1 & -3 \\ 3 & -5 \end{bmatrix} \)

(b) \( A = \begin{bmatrix} 3 & -1 & 4 \\ -1 & 0 & 1 \\ 4 & 1 & 2 \end{bmatrix} \)

(c) \( A = \begin{bmatrix} 2 & -2 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \)

(d) \( A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \)

**ML.3.** In each of the following cases, \( \lambda \) is an eigenvalue of A. Use MATLAB to find a corresponding eigenvector.

(a) \( \lambda = 3, A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \)

(b) \( \lambda = -1, A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & 0 \\ 2 & 1 & -1 \end{bmatrix} \)

(c) \( \lambda = 2, A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix} \)

**ML.4.** Consider a living organism that can live to a maximum age of two years and whose Leslie matrix is

\[
\begin{bmatrix}
0.2 & 0.8 & 0.3 \\
0.9 & 0 & 0 \\
0 & 0.7 & 0
\end{bmatrix}
\]

Find a stable age distribution.

### 8.2 Diagonalization

In this section we show how to find the eigenvalues and associated eigenvectors of a given matrix \( A \) by finding the eigenvalues and eigenvectors of a related matrix \( B \) that has the same eigenvalues and eigenvectors as \( A \). The matrix \( B \) has the helpful property that its eigenvalues are easily obtained. Thus, we will have found the eigenvalues of \( A \). In Section 8.3, this approach will shed much light on the eigenvalue-eigenvector problem. For convenience, we only work with matrices all of whose entries and eigenvalues are real numbers.

**Similar Matrices**

A matrix \( B \) is said to be similar to a matrix \( A \) if there is a nonsingular matrix \( P \) such that

\[
B = P^{-1}AP.
\]
Key Terms
Similar matrices
Diagonalizable
Diagonlized
Distinct eigenvalues
Multiplicity of an eigenvalue
Defective matrix

8.2 Exercises

In Exercises 1 through 8, determine whether the given matrix is diagonalizable.

1. \[
\begin{bmatrix}
1 & 4 \\
1 & -2
\end{bmatrix}
\]
2. \[
\begin{bmatrix}
1 & 0 \\
-2 & 1
\end{bmatrix}
\]
3. \[
\begin{bmatrix}
1 & 1 & -2 \\
4 & 0 & 4 \\
1 & -1 & 4
\end{bmatrix}
\]
4. \[
\begin{bmatrix}
1 & 2 & 3 \\
0 & -1 & 2 \\
0 & 0 & 2
\end{bmatrix}
\]
5. \[
\begin{bmatrix}
3 & 1 & 0 \\
0 & 3 & 1 \\
0 & 0 & 3
\end{bmatrix}
\]
6. \[
\begin{bmatrix}
-2 & 2 \\
5 & 1
\end{bmatrix}
\]
7. \[
\begin{bmatrix}
2 & 0 & 3 \\
0 & 1 & 0 \\
0 & 1 & 2
\end{bmatrix}
\]
8. \[
\begin{bmatrix}
2 & 3 & 3 & 5 \\
3 & 2 & 2 & 3 \\
0 & 0 & 2 & 2 \\
0 & 0 & 0 & 2
\end{bmatrix}
\]

9. Find a 2 x 2 nondiagonal matrix whose eigenvalues are 2 and -3, and associated eigenvectors are
\[
\begin{bmatrix}
-1 \\
2
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
1 \\
1
\end{bmatrix}
\]
respectively.

10. Find a 3 x 3 nondiagonal matrix whose eigenvalues are -2, -2, and 3, and associated eigenvectors are
\[
\begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix},
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]
respectively.

In Exercises 11 through 22, find, if possible, a nonsingular matrix P such that \(P^{-1}AP\) is diagonal.

11. \[
\begin{bmatrix}
4 & 2 & 3 \\
2 & 1 & 2 \\
-1 & -2 & 0
\end{bmatrix}
\]
12. \[
\begin{bmatrix}
1 & 1 & 2 \\
0 & 1 & 0 \\
0 & 1 & 3
\end{bmatrix}
\]
13. \[
\begin{bmatrix}
1 & 2 & 3 \\
0 & 1 & 0 \\
2 & 1 & 2
\end{bmatrix}
\]
14. \[
\begin{bmatrix}
0 & -1 \\
2 & 3
\end{bmatrix}
\]
15. \[
\begin{bmatrix}
8 & 1 & 0 \\
0 & 8 & 0 \\
8 & 0 & 0
\end{bmatrix}
\]
16. \[
\begin{bmatrix}
3 & 0 & 0 \\
1 & 3 & 1 \\
0 & 0 & 1
\end{bmatrix}
\]
17. \[
\begin{bmatrix}
3 & -2 & 1 \\
0 & 2 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
18. \[
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 2 \\
2 & 2 & 2
\end{bmatrix}
\]
19. \[
\begin{bmatrix}
3 & 0 & 0 \\
2 & 3 & 0 \\
0 & 0 & 3
\end{bmatrix}
\]
20. \[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 0 & 3 \\
0 & 1 & 2
\end{bmatrix}
\]
21. \[
\begin{bmatrix}
4 & 1 & 0 & 0 \\
0 & 4 & 0 & 0 \\
0 & 0 & 4 & 0 \\
1 & 1 & 0 & 0
\end{bmatrix}
\]
22. \[
\begin{bmatrix}
3 & 0 & 0 & 0 \\
2 & 3 & 0 & 0 \\
0 & 0 & 5 & 1 \\
0 & 0 & 0 & 6
\end{bmatrix}
\]
23. Let A be a 2 x 2 matrix whose eigenvalues are 3 and 4, and associated eigenvectors are
\[
\begin{bmatrix}
-1 \\
1
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
2 \\
1
\end{bmatrix}
\]
respectively. Without computation, find a diagonal matrix D that is similar to A and nonsingular matrix P such that \(P^{-1}AP = D\).

24. Let A be a 3 x 3 matrix whose eigenvalues are -3, 4, and 4, and associated eigenvectors are
\[
\begin{bmatrix}
-1 \\
0 \\
1
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix}
\]
respectively. Without computation, find a diagonal matrix D that is similar to A and nonsingular matrix P such that \(P^{-1}AP = D\).

In Exercises 25 through 28, find two matrices that are similar to A.

25. \[
A = \begin{bmatrix}
3 & 4 \\
0 & 0
\end{bmatrix}
\]
26. \[
A = \begin{bmatrix}
1 & 2 \\
3 & 0
\end{bmatrix}
\]
27. \[
A = \begin{bmatrix}
2 & 0 & 0 \\
1 & 1 & 0 \\
-1 & 0 & 1
\end{bmatrix}
\]
28. \[
A = \begin{bmatrix}
2 & 0 & 0 \\
1 & 2 & 1 \\
0 & 1 & 2
\end{bmatrix}
\]
In Exercises 29 through 32, determine whether the given matrix is similar to a diagonal matrix.

29. \[
\begin{bmatrix}
2 & 3 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2 \\
\end{bmatrix}
\]

30. \[
\begin{bmatrix}
2 & 3 & 1 \\
0 & 1 & 0 \\
0 & 0 & 2 \\
\end{bmatrix}
\]

31. \[
\begin{bmatrix}
-3 & 0 \\
1 & 2 \\
\end{bmatrix}
\]

32. \[
\begin{bmatrix}
1 & 1 & 0 \\
2 & 2 & 0 \\
3 & 3 & 3 \\
\end{bmatrix}
\]

In Exercises 33 through 36, show that each matrix is diagonalizable and find a diagonal matrix similar to the given matrix.

33. \[
\begin{bmatrix}
4 & 2 \\
3 & 3 \\
\end{bmatrix}
\]

34. \[
\begin{bmatrix}
3 & 2 \\
6 & 4 \\
\end{bmatrix}
\]

35. \[
\begin{bmatrix}
2 & -2 & 3 \\
0 & 3 & -2 \\
0 & -1 & 2 \\
\end{bmatrix}
\]

36. \[
\begin{bmatrix}
0 & -2 & 1 \\
1 & 3 & -1 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

In Exercises 37 through 40, show that the given matrix is not diagonalizable.

37. \[
\begin{bmatrix}
1 & 1 \\
0 & 1 \\
\end{bmatrix}
\]

38. \[
\begin{bmatrix}
2 & 0 & 0 \\
3 & 2 & 0 \\
0 & 0 & 5 \\
\end{bmatrix}
\]

39. \[
\begin{bmatrix}
10 & 11 & 3 \\
-3 & -4 & -3 \\
-8 & -8 & -1 \\
\end{bmatrix}
\]

40. \[
\begin{bmatrix}
2 & 3 & 3 & 5 \\
3 & 2 & 2 & 3 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

A matrix \(A\) is called defective if \(A\) has an eigenvalue \(\lambda\) of multiplicity \(m > 1\) for which the associated eigenspace has a basis of fewer than \(m\) vectors; that is, the dimension of the eigenspace associated with \(\lambda\) is less than \(m\). In Exercises 41 through 44, use the eigenvalues of the given matrix to determine if the matrix is defective.

41. \[
\begin{bmatrix}
8 & 7 \\
0 & 8 \\
\end{bmatrix}
\]

\(\lambda = 8, 8\)

42. \[
\begin{bmatrix}
3 & 0 & 0 \\
-2 & 3 & -2 \\
2 & 0 & 5 \\
\end{bmatrix}
\]

\(\lambda = 3, 3, 5\)

43. \[
\begin{bmatrix}
3 & 3 & 3 \\
3 & 3 & 3 \\
-3 & -3 & -3 \\
\end{bmatrix}
\]

\(\lambda = 0, 0, 3\)

44. \[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
\end{bmatrix}
\]

\(\lambda = 1, 1, -1, -1\)

45. Let \(D = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}\). Compute \(D^9\).

46. Let \(A = \begin{bmatrix} 3 & -5 \\ 1 & -3 \end{bmatrix}\). Compute \(A^9\). (Hint: Find a matrix \(P\) such that \(P^{-1}AP\) is a diagonal matrix \(D\) and show that \(A^9 = PDP^9P^{-1}\)).

Theoretical Exercises

T.1. Show that:
(a) \(A\) is similar to \(A\).
(b) If \(B\) is similar to \(A\), then \(A\) is similar to \(B\).
(c) If \(A\) is similar to \(B\) and \(B\) is similar to \(C\), then \(A\) is similar to \(C\).

T.2. Show that if \(A\) is nonsingular and diagonalizable, then \(A^{-1}\) is diagonalizable.

T.3. Let \(A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\). Find necessary and sufficient conditions for \(A\) to be diagonalizable.

T.4. Let \(A\) and \(B\) be nonsingular \(n \times n\) matrices. Show that \(AB^{-1}\) and \(B^{-1}A\) have the same eigenvalues.

T.5. Prove or disprove: Every nonsingular matrix is similar to a diagonal matrix.

T.6. If \(A\) and \(B\) are nonsingular, show that \(AB\) and \(BA\) are similar.

T.7. Show that if \(A\) is diagonalizable, then:
(a) \(A^T\) is diagonalizable.
(b) \(A^k\) is diagonalizable, where \(k\) is a positive integer.

T.8. Show that if \(A\) and \(B\) are similar matrices, then \(A^k\) and \(B^k\), for any nonnegative integer \(k\), are similar.

T.9. Show that if \(A\) and \(B\) are similar matrices, then \(\text{det}(A) = \text{det}(B)\).

T.10. Let \(A\) be an \(n \times n\) matrix and let \(B = P^{-1}AP\) be similar to \(A\). Show that if \(x\) is an eigenvector of \(A\) associated with the eigenvalue \(\lambda\) of \(A\), then \(P^{-1}x\) is an eigenvector of \(B\) associated with the eigenvalue \(\lambda\) of the matrix \(B\).

T.11. Let \(\lambda_1, \lambda_2, \ldots, \lambda_k\) be distinct eigenvalues of a matrix \(A\) with associated eigenvectors \(x_1, x_2, \ldots, x_k\). Show that \(x_1, x_2, \ldots, x_k\) are linearly independent. (Hint: See the proof of Theorem 8.5.)

T.12. Show that if \(A\) and \(B\) are similar matrices, then they have the same characteristic polynomial.
MATLAB Exercises

ML.1. Use MATLAB to determine if $A$ is diagonalizable. If it is, find a nonsingular matrix $P$ so that $P^{-1}AP$ is diagonal.

(a) $A = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & -3 \\ 3 & -5 \end{bmatrix}$

(c) $A = \begin{bmatrix} 0 & 0 & 4 \\ 5 & 3 & 6 \\ 6 & 0 & 5 \end{bmatrix}$

ML.2. Use MATLAB and the hint in Exercise 46 to compute $A^{30}$, where

$$A = \begin{bmatrix} -1 & 1 & -1 \\ -2 & 2 & -1 \\ -2 & 2 & -1 \end{bmatrix}.$$  

ML.3. Repeat Exercise ML.2 for

$$A = \begin{bmatrix} -1 & 1.5 & -1.5 \\ -2 & 2.5 & -1.5 \\ -2 & 2.0 & -1.0 \end{bmatrix}.$$  

Display your answer in both format short and format long.

ML.4. Use MATLAB to investigate the sequences $A, A^3, A^5, \ldots$ and $A^2, A^4, A^6, \ldots$ for matrix $A$ in Exercise ML.2. Write a brief description of the behavior of these sequences. Describe $\lim_{n \to \infty} A^n$.

8.3 DIAGONALIZATION OF SYMMETRIC MATRICES

In this section we consider the diagonalization of symmetric matrices (i.e., $n \times n$ matrix $A$ with real entries such that $A = A^T$). We restrict our attention to this case because it is easier to handle than that of general matrices and also because symmetric matrices arise in many applied problems.

As an example of such a problem, consider the task of identifying the conic represented by the equation

$$2x^2 + 2xy + 2y^2 = 9,$$

which can be written in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 9.$$  

Observe that the matrix used here is a symmetric matrix. This problem is discussed in detail in Section 9.5. We shall merely remark here that the solution calls for the determination of the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$  

The $x$- and $y$-axes are then rotated to a new set of axes, which lie along the eigenvectors of the matrix. In the new set of axes, the given conic can be identified readily.

We omit the proof of the following important theorem (see D. R. Hill, Experiments in Computational Matrix Algebra, New York: Random House, 1988).

**Theorem 8.6**

All the roots of the characteristic polynomial of a symmetric matrix are real numbers.  

**Theorem 8.7**

If $A$ is a symmetric matrix, then eigenvectors that are associated with distinct eigenvalues of $A$ are orthogonal.
We now provide a summary of properties of eigenvalues and eigenvectors.

**SUMMARY OF PROPERTIES OF EIGENVALUES AND EIGENVECTORS**

Let $A$ be an $n \times n$ matrix.

- If $x$ and $y$ are eigenvectors associated with the eigenvalue $\lambda$ of $A$, then if $x + y \neq 0$, it follows that $x + y$ is also an eigenvector associated with $\lambda$.
- If $x$ is an eigenvector associated with the eigenvalue $\lambda$ of $A$, then $kx$, $k \neq 0$, is also an eigenvector associated with $\lambda$.
- If $\lambda$ is an eigenvalue of $A$ and $x$ is an associated eigenvector, then for any nonnegative integer $k$, $\lambda^k$ is an eigenvalue of $A^k$ and $x$ is an associated eigenvector.
- If $\lambda$ and $\mu$ are distinct eigenvalues of $A$ with associated eigenvectors $x$ and $y$, respectively, then $x$ and $y$ are linearly independent. That is, eigenvectors that are associated with distinct eigenvalues are linearly independent.
- $A$ and $A^T$ have the same eigenvalues.
- If $A$ is a diagonal, upper triangular, or lower triangular matrix, then its eigenvalues are the entries on its main diagonal.
- The eigenvalues of a symmetric matrix are all real.
- Eigenvectors associated with distinct eigenvalues of a symmetric matrix are orthogonal.
- $\det(A)$ is the product of all the roots of the characteristic polynomial of $A$. [Equivalently, $\det(A)$ is the product of the eigenvalues of $A$.]
- $A$ is singular if and only if 0 is an eigenvalue of $A$.
- Similar matrices have the same eigenvalues.
- $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors.
- If $A$ has $n$ distinct eigenvalues, then $A$ is diagonalizable.

**Key Terms**

Orthogonal matrix
Orthogonally diagonalizable
Jordan canonical form

### 8.3 Exercises

1. Verify that

$$ P = \begin{bmatrix} \frac{1}{3} & -\frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{3} & 2 & \frac{1}{3} \end{bmatrix} $$

is an orthogonal matrix.

2. Find the inverse of each of the following orthogonal matrices.

(a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

3. Verify Theorem 8.8 for the matrices in Exercise 2.

4. Verify that the matrix $P$ in Example 3 is an orthogonal matrix.

In Exercises 5 through 10, orthogonally diagonalize each given matrix $A$, giving the diagonal matrix $D$ and the
diagonalizing orthogonal matrix $P$. 

5. $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$  
6. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$  
7. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$  
8. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$  
9. $\begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$  
10. $\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$  
11. $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$  
12. $\begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$  
13. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  
14. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{bmatrix}$  
15. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$  
16. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$  
17. $\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$  
18. $\begin{bmatrix} -3 & 0 & -1 \\ 0 & -2 & 0 \\ -1 & 0 & -3 \end{bmatrix}$ 

In Exercises 11 through 18, orthogonally diagonalize each given matrix.

Theoretical Exercises

T.1. Show that if $x$ and $y$ are vectors in $R^n$, then 
$(Ax) \cdot y = x \cdot (A^T y)$.

T.2. Show that if $A$ is an $n \times n$ orthogonal matrix and $x$ and $y$ are vectors in $R^n$, then $(Ax) \cdot (Ay) = x \cdot y$.


T.4. Show that if $A$ is an orthogonal matrix, then $
\det(A) = \pm 1$.

T.5. Prove Theorem 8.9 for the $2 \times 2$ case by studying the possible roots of the characteristic polynomial of $A$.

T.6. Show that if $A$ and $B$ are orthogonal matrices, then $AB$ is an orthogonal matrix.

T.7. Show that if $A$ is an orthogonal matrix, then $A^{-1}$ is also orthogonal.

T.8. (a) Verify that the matrix

$$
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta 
\end{bmatrix}
$$

is orthogonal.

(b) Show that if $A$ is an orthogonal $2 \times 2$ matrix, then there exists a real number $\theta$ such that either

$$
A = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta 
\end{bmatrix}
$$

or

$$
A = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta 
\end{bmatrix}
$$

T.9. Show that if $A^T A y = y$ for any $y$ in $R^n$, then $A^T A = I_n$.

T.10. Show that if $A$ is nonsingular and orthogonally diagonalizable, then $A^{-1}$ is orthogonally diagonalizable.

MATLAB Exercises

The MATLAB command eig will produce the eigenvalues and a set of orthonormal eigenvectors for a symmetric matrix $A$. Use the command in the form

$$
[V, D] = \text{eig}(A)
$$

The matrix $V$ will contain the orthonormal eigenvectors, and matrix $D$ will be diagonal with the corresponding eigenvalues on the main diagonal.

ML.1. Use eig to find the eigenvalues of $A$ and an orthogonal matrix $P$ so that $P^T A P$ is diagonal.

(a) $A = \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$

ML.2. Command eig can be applied to any matrix, but the matrix $V$ of eigenvectors need not be orthogonal. For each of the following, use eig to determine which matrices $A$ are such that $V$ is orthogonal. If $V$ is not orthogonal, then discuss briefly whether it can or cannot be replaced by an orthogonal matrix of eigenvectors.
Key Ideas for Review

- **Theorem 8.2.** The eigenvalues of $A$ are the roots of the characteristic polynomial of $A$.
- **Theorem 8.3.** Similar matrices have the same eigenvalues.
- **Theorem 8.4.** An $n \times n$ matrix $A$ is diagonalizable if and only if it has $n$ linearly independent eigenvectors. In this case $A$ is similar to a diagonal matrix $D$, with $D = P^{-1}AP$, whose diagonal elements are the eigenvalues of $A$, and $P$ is a matrix whose columns are $n$ linearly independent eigenvectors of $A$.
- **Theorem 8.6.** All the roots of the characteristic polynomial of a symmetric matrix are real numbers.
- **Theorem 8.7.** If $A$ is a symmetric matrix, then eigenvectors that belong to distinct eigenvalues of $A$ are orthogonal.
- **Theorem 8.8.** The $n \times n$ matrix $A$ is orthogonal if and only if the columns of $A$ form an orthonormal set of vectors in $\mathbb{R}^n$.
- **Theorem 8.9.** If $A$ is a symmetric $n \times n$ matrix, then there exists an orthogonal matrix $P$ ($P^{-1} = P^T$) such that $P^{-1}AP = D$, a diagonal matrix. The eigenvalues of $A$ lie on the main diagonal of $D$.
- **List of Nonsingular Equivalences.** The following statements are equivalent for an $n \times n$ matrix $A$.
  1. $A$ is nonsingular.
  2. $x = 0$ is the only solution to $Ax = 0$.
  3. $A$ is row equivalent to $I_n$.
  4. The linear system $Ax = b$ has a unique solution for every $n \times 1$ matrix $b$.
  5. $\det(A) \neq 0$.
  6. $A$ has rank $n$.
  7. $A$ has nullity 0.
  8. The rows of $A$ form a linearly independent set of $n$ vectors in $\mathbb{R}^n$.
  9. The columns of $A$ form a linearly independent set of $n$ vectors in $\mathbb{R}^n$.
  10. Zero is not an eigenvalue of $A$.

Supplementary Exercises

1. Find the characteristic polynomial, eigenvalues, and eigenvectors of the matrix

$$
\begin{bmatrix}
-2 & 0 & 0 \\
3 & 2 & 3 \\
4 & -1 & 6
\end{bmatrix}.
$$

In Exercises 2 and 3, determine whether the given matrix is diagonalizable.

2. $A = \begin{bmatrix} -1 & 0 & 0 \\ 3 & 2 & 0 \\ 4 & -1 & 2 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & 2 & 0 \\ 5 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & -6 \\ 2 & 0 & -2 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

6. If possible, find a diagonal matrix $D$ so that

$$
A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}
$$
is similar to $D$.

7. Find bases for the eigenspaces associated with each eigenvalue of the matrix

$$
A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.
$$

8. Is the matrix

$$
\begin{bmatrix}
\frac{2}{3} & \frac{1}{\sqrt{3}} & 1 \\
\frac{2}{3} & 0 & 0 \\
\frac{1}{3} & -\frac{2}{\sqrt{3}} & 0
\end{bmatrix}
$$

orthogonal?

In Exercises 9 and 10, orthogonally diagonalize the given matrix $A$, giving the orthogonal matrix $P$ and the diagonal matrix $D$.

9. $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

10. $A = \begin{bmatrix} -3 & 0 & -4 \\ 0 & 5 & 0 \\ -4 & 0 & 3 \end{bmatrix}$
### 7.1 Exercises

In Exercises 1 through 6, compute the QR-factorization of $A$.

1. $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$
2. $A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \\ 1 & 1 \end{bmatrix}$
3. $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -3 & 3 \\ -1 & 2 & 4 \end{bmatrix}$
4. $A = \begin{bmatrix} 2 & -1 \\ -1 & 3 \\ 0 & 1 \end{bmatrix}$
5. $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ -1 & -2 & 2 \end{bmatrix}$
6. $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

### Theoretical Exercises

**T.1.** In the proof of Theorem 7.1, show that $r_{ii}$ is nonzero by first expressing $u_i$ as a linear combination of $v_1, v_2, \ldots, v_i$ and then computing $r_{ii} = u_i \cdot w_i$.

**T.2.** Show that every nonsingular matrix has a QR-factorization.

### 7.2 LEAST SQUARES

**Prerequisites.** Section 1.6, Solutions of Linear Systems of Equations, Section 1.7, The Inverse of a Matrix, Section 4.2, $n$-Vectors, Section 6.9, Orthogonal Complements.

From Chapter 1 we recall that an $m \times n$ linear system $Ax = b$ is inconsistent if it has no solution. In the proof of Theorem 6.14 in Section 6.6 we show that $Ax = b$ is consistent if and only if $b$ belongs to the column space of $A$. Equivalently, $Ax = b$ is inconsistent if and only if $b$ is not in the column space of $A$. Inconsistent systems do indeed arise in many situations and we must determine how to deal with them. Our approach is to change the problem so that we do not require that the matrix equation $Ax = b$ be satisfied. Instead, we seek a vector $\tilde{x}$ in $\mathbb{R}^n$ such that $A\tilde{x}$ is as close to $b$ as possible. If $W$ is the column space of $A$, then from Theorem 6.23 in Section 6.9, it follows that the vector in $W$ that is closest to $b$ is $\text{proj}_W b$. That is, $\|b - w\|$ is minimized when $w = \text{proj}_W b$. Thus, if we find $\tilde{x}$ such that $A\tilde{x} = \text{proj}_W b$ then we are assured that $\|b - A\tilde{x}\|$ will be as small as possible. As shown in the proof of Theorem 6.23, $b - \text{proj}_W b = b - A\tilde{x}$ is orthogonal to every vector in $W$. (See Figure 7.1.) It then follows that $b - A\tilde{x}$ is orthogonal to each column of $A$. In terms of a matrix equation, we have

$$A^T (A\tilde{x} - b) = 0$$

or, equivalently,

$$A^T A\tilde{x} = A^T b.$$  

Thus, $\tilde{x}$ is a solution to

$$A^T Ax = A^T b.$$
7.2 Exercises

In Exercises 1 through 4, find a least squares solution to \( Ax = b \).

1. \( A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} \) and \( b = \begin{bmatrix} 3 \\ 1 \\ -2 \\ -1 \end{bmatrix} \)

2. \( A = \begin{bmatrix} 3 & -2 \\ 2 & -3 \\ 1 & -1 \\ 2 & 3 \end{bmatrix} \) and \( b = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \)

3. \( A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 5 & 3 \\ 2 & 0 & 1 \end{bmatrix} \) and \( b = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \)

4. \( A = \begin{bmatrix} -1 & 0 & 0 & 4 \\ 4 & -2 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 2 & -2 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \) and \( b = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \)

5. Solve Exercise 1 using the QR-factorization of \( A \).

6. Solve Exercise 3 using the QR-factorization of \( A \).

In Exercises 7 through 10, find the least squares line for the given data points.

7. \((3, 1), (3, 2), (4, 3), (5, 2)\)

8. \((3, 2), (4, 3), (5, 2), (6, 4), (7, 3)\)

9. \((2, 3), (3, 4), (4, 3), (5, 4), (6, 3), (7, 4)\)

10. \((3, 3), (4, 5), (5, 4), (6, 5), (7, 5), (8, 6), (9, 5), (10, 6)\)

In Exercises 11 and 12, find a quadratic least squares polynomial for the given data.

11. \((0, 3.2), (0.5, 1.6), (1, 2), (2, -0.4), (2.5, -0.8), (3, -1.6), (4, 0.3), (5, 2.2)\)

12. \((0.5, -1.6), (1, 0.4), (1.5, 0.7), (2, 1.8), (2.5, 1.6), (3, 2.2), (3.5, 1.7), (4, 2.2), (4.5, 1.6), (5, 1.5)\)

13. In an experiment designed to determine the extent of a person's natural orientation, a subject is put in a special room and kept there for a certain length of time. He is then asked to find a way out of a maze and a record is made of the time it takes the subject to accomplish this task. The following data are obtained:

<table>
<thead>
<tr>
<th>Time in Room (hours)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time to Find Way Out of Maze (minutes)</td>
<td>0.8</td>
<td>2.1</td>
<td>2.6</td>
<td>2.0</td>
<td>3.1</td>
<td>3.3</td>
</tr>
</tbody>
</table>

Let \( x \) denote the number of hours in the room and let \( y \) denote the number of minutes that it takes the subject to find his way out.

(a) Find the least squares line relating \( x \) and \( y \).

(b) Use the equation obtained in (a) to estimate the time it will take the subject to find his way out of the maze after 10 hours in the room.

14. A steel producer gathers the following data.

<table>
<thead>
<tr>
<th>Year</th>
<th>1997</th>
<th>1998</th>
<th>1999</th>
<th>2000</th>
<th>2001</th>
<th>2002</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annual Sales (millions of dollars)</td>
<td>1.2</td>
<td>2.3</td>
<td>3.2</td>
<td>3.6</td>
<td>3.8</td>
<td>5.1</td>
</tr>
</tbody>
</table>

Represent the years 1997, \ldots, 2002 as 0, 1, 2, 3, 4, 5, respectively, and let \( x \) denote the year. Let \( y \) denote the annual sales (in millions of dollars).

(a) Find the least squares line relating \( x \) and \( y \).

(b) Use the equation obtained in (a) to estimate the annual sales for the year 2006.

15. A sales organization obtains the following data relating the number of salespersons to annual sales.

<table>
<thead>
<tr>
<th>Number of Salespersons</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annual Sales (millions of dollars)</td>
<td>2.3</td>
<td>3.2</td>
<td>4.1</td>
<td>5.0</td>
<td>6.1</td>
<td>7.2</td>
</tr>
</tbody>
</table>

Let \( x \) denote the number of salespersons and let \( y \) denote the annual sales (in millions of dollars).

(a) Find the least squares line relating \( x \) and \( y \).

(b) Use the equation obtained in (a) to estimate the annual sales when there are 14 salespersons.
16. The distributor of a new car has obtained the following data.

<table>
<thead>
<tr>
<th>Number of Weeks After Introduction of Car</th>
<th>Gross Receipts per Week (millions of dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.8</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
</tr>
<tr>
<td>3</td>
<td>3.2</td>
</tr>
<tr>
<td>4</td>
<td>4.3</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>5.1</td>
</tr>
<tr>
<td>7</td>
<td>4.3</td>
</tr>
<tr>
<td>8</td>
<td>3.8</td>
</tr>
<tr>
<td>9</td>
<td>1.2</td>
</tr>
<tr>
<td>10</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Let $x$ denote the gross receipts per week (in millions of dollars) $t$ weeks after the introduction of the car.

(a) Find a least squares quadratic polynomial for the given data.

(b) Use the equation in part (a) to estimate the gross receipts 12 weeks after the introduction of the car.

17. Given $Ax = b$, where

$$A = \begin{bmatrix} 1 & 3 & -3 \\ 2 & 4 & -2 \\ 0 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$ 

(a) Show that rank $A = 2$.

(b) Since rank $A \neq$ number of columns, Theorem 7.2 cannot be used to determine a least squares solution $\hat{x}$. Follow the general procedure as discussed prior to Theorem 7.2 to find a least squares solution. Is the solution unique?

Theoretical Exercises

T1. Suppose that we wish to find the least squares line for the $n$ data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, so that $m = 1$ in (6). Show that if at least two $x$-coordinates are unequal, then the matrix $A^TA$ is nonsingular, where $A$ is the matrix resulting from the $n$ equations in (6).

MATLAB Exercises

Routine Isqline in MATLAB will compute the least squares line for data you supply and graph both the line and the data points. To use Isqline, put the $x$-coordinates of your data into a vector $\mathbf{x}$ and the corresponding $y$-coordinates into a vector $\mathbf{y}$ and then type Isqline($\mathbf{x, y}$). For more information, use help Isqline.

ML1. Solve Exercise 9 in MATLAB using Isqline.

ML2. Use Isqline to determine the solution to Exercise 13. Then estimate the time it will take the subject to find his way out of the maze after 7 hours, 8 hours, 9 hours.

ML3. An experiment was conducted on the temperatures of a fluid in a newly designed container. The following data were obtained.

<table>
<thead>
<tr>
<th>Time (minutes)</th>
<th>0</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature (°F)</td>
<td>185</td>
<td>170</td>
<td>166</td>
<td>152</td>
<td>110</td>
</tr>
</tbody>
</table>

(a) Determine the least squares line.

(b) Estimate the temperature at $x = 1, 6, 8$ minutes.

(c) Estimate the time at which the temperature of the fluid was 160°F.

ML4. At time $t = 0$ an object is dropped from a height of 1 meter above a fluid. A recording device registers the height of the object above the surface of the fluid at $\frac{1}{2}$ second intervals, with a negative value indicating the object is below the surface of the fluid. The following table of data is the result.

<table>
<thead>
<tr>
<th>Time (seconds)</th>
<th>Depth (meters)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0.5</td>
<td>0.88</td>
</tr>
<tr>
<td>1</td>
<td>0.54</td>
</tr>
<tr>
<td>1.5</td>
<td>0.07</td>
</tr>
<tr>
<td>2</td>
<td>-0.42</td>
</tr>
<tr>
<td>2.5</td>
<td>-0.80</td>
</tr>
<tr>
<td>3</td>
<td>-0.99</td>
</tr>
<tr>
<td>3.5</td>
<td>-0.94</td>
</tr>
<tr>
<td>4</td>
<td>-0.65</td>
</tr>
<tr>
<td>4.5</td>
<td>-0.21</td>
</tr>
</tbody>
</table>

(a) Determine the least squares quadratic polynomial.

(b) Estimate the depth at $t = 5$ and $t = 6$ seconds.

(c) Estimate the time the object breaks through the surface of the fluid the second time.
ML.5. Determine the least squares quadratic polynomial for the following table of data. Use this data model to predict the value of \( y \) when \( x = 7 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>0.5</td>
</tr>
<tr>
<td>-2.5</td>
<td>0</td>
</tr>
<tr>
<td>-2</td>
<td>-1.125</td>
</tr>
<tr>
<td>-1.5</td>
<td>-1.875</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>0.9375</td>
</tr>
<tr>
<td>0.5</td>
<td>2.8750</td>
</tr>
<tr>
<td>1</td>
<td>4.75</td>
</tr>
<tr>
<td>1.5</td>
<td>8.25</td>
</tr>
<tr>
<td>2</td>
<td>11.5</td>
</tr>
</tbody>
</table>

### 7.3 MORE ON CODING*

**Prerequisite.** Section 2.1, An Introduction to Coding

In Section 2.1, An Introduction to Coding, we briefly discussed binary codes and error detection. In this section we show how the techniques of linear algebra can be used to develop error-correcting binary codes. We stress the role that matrices, vector spaces, and associated concepts play in constructing some simple codes and cite several references at the end of the section for a more in-depth study of the fascinating area of coding theory.

The examples and exercises in Section 2.1 contained codes generated by a coding function \( e : B^m \rightarrow B^n \), where \( n > m \) and \( e \) is one-to-one. In this section we approach the generation of codes from a related point of view. Here we will generate binary codes using a matrix transformation from \( B^m \) to \( B^n \), and will choose the matrix of the transformation in such a way that we will be guaranteed that there is a one-to-one correspondence between the message vector from \( B^m \) and the code word in \( B^n \). For a message vector \( b \) in \( B^m \), define the matrix transformation

\[
e(b) = Cb,
\]

where \( C \) is the \( n \times m \) matrix

\[
\begin{bmatrix}
I_m \\
D
\end{bmatrix}
\]

and \( D \) is an \((n - m) \times m\) matrix. Each code word in \( B^n \) has the form

\[
Cb = \begin{bmatrix}
l_m \\
D
\end{bmatrix}b = \begin{bmatrix}
b \\
Db
\end{bmatrix},
\]

where \( Db \) is an \((n - m) \times 1\) vector. The matrix \( D \) will be chosen to aid in error detection and correction.

**Theorem 7.3**

The matrix transformation \( e : B^m \rightarrow B^n \) defined by

\[
e(b) = Cb = \begin{bmatrix}
l_m \\
D
\end{bmatrix}b, \quad \text{is one-to-one.}
\]

where \( b \) is an \( m \)-vector and \( D \) is an \((n - m) \times m\) matrix.

*Throughout this section only binary arithmetic is used.*