### Definitions

**matrix**

An $n \times m$ matrix $A = (a_{ij})$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ is a rectangular array of numbers

$$A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1m} \\
a_{21} & a_{22} & \cdots & a_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nm}
\end{bmatrix}$$

**transpose**

The transpose of a matrix $A = (a_{ij})$ is $A^T = (a_{ji})$, where $a_{ji}$ is the element in the $i$-th row and $j$-th column.

**graph, adjacency matrix**

A graph is a collection of edges and nodes. The adjacency matrix of a graph is a matrix $A = (a_{ij})$ where $a_{ij}$ represents the number of links between nodes $i$ and $j$.

**diagonal**

The diagonal of a matrix $A = (a_{ij})$ is the set $\{a_{ii} : i = 1, \ldots, n\}$

**diagonal matrix**

A diagonal matrix is a matrix $A = (a_{ij})$ such that $a_{ij} = 0$ if $i \neq j$.

**identity matrix**

The identity matrix is a square matrix $I = (s_{ij})$ where $s_{ij}$ is the Kronecker delta function $s_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

**trace**

The trace of $A = (a_{ij})$ is the sum of the diagonal elements $\text{tr} A = \sum_i a_{ii}$

### Theorems/Formulas

Let $A$ be a matrix and $\mathbf{x}$ be column vector. If $A\mathbf{x} = \mathbf{0}$, then the vector $\mathbf{x}$ is orthogonal to the rows of $A$.

$$(AB)^T = B^T A^T$$

Let $A, B$ be matrices such that $AB$ makes sense. Then $(AB)^T = B^T A^T$.

$$\text{tr}(AB) = \text{tr}(BA)$$

For any square matrices $A$ and $B$, $\text{tr}(AB) = \text{tr}(BA)$

### Properties of the Inverse

1. $(A^T)^T = A$
2. $(A B)^T = B^T A^T$
3. $(A^T)^T = (A^T)^T$
4. $(A^T)^T = (A)^T$
### Definitions

**Permutation**

A permutation is the shuffling of $n$ elements. A permutation is even/odd if the number of swaps is even/odd.

**Sign Function**

The sign function is defined by $\text{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ \text{-}1 & \text{if } \sigma \text{ is odd} \end{cases}$

**Determinant**

The determinant of an $n \times n$ matrix $A = (a_{ij})$ is $\det(A) = \sum_{\sigma} \text{sgn}(\sigma)a_{1\sigma(1)}a_{2\sigma(2)} \ldots a_{n\sigma(n)}$.

### Elementary Row Matrices

**Minor, Cofactor**

Let $A = (a_{ij})$. The minor of $a_{ij}$ is the determinant of $A$ without the $i$th row and $j$th column. The cofactor of $a_{ij}$ is $(-1)^{i+j} \text{minor}(a_{ij})$.

**Adjoint/Adjugate Matrix**

Let $A = (a_{ij})$ be a square matrix. The adjoint (or adjugate) matrix of $A$ is given by $\text{adj } A = (\text{cofactor}(a_{ij}))^T$.

### Theorems/Formulas

If a square matrix $A$ has a row of all zeroes, then $\det(A) = 0$.

**Determinants of Elementary Row Matrices**

- $\det(P_{ij}) = -1$
- $\det(R_i(\lambda)) = \lambda$
- $\det(E_{ij}(\mu)) = 1$

**Properties of the Determinant**

- $\det(AB) = \det(A) \det(B)$
- $\det(A^T) = \det(A)$
- $\det(A^T) = (\det(A))^T$

If $A$ is invertible, then $A^{-1} = \frac{1}{\det A} \text{adj } A$. 

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<thead>
<tr>
<th>Definitions</th>
<th>Elementary Row Matrices</th>
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<tbody>
<tr>
<td><strong>Permutation</strong></td>
<td>P$_{ij}$ swaps rows i and j</td>
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<tr>
<td><strong>Sign Function</strong></td>
<td>R$_i(\lambda)$ multiplies row i by a scalar $\lambda$</td>
</tr>
<tr>
<td><strong>Determinant</strong></td>
<td>E$_{ij}(\mu)$ adds $\mu$ times row j to row i</td>
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### Definitions

**subspace**
A subset $U$ of a vector space $V$ is a subspace of $V$ if $U$ is also a vector space.

**span**
Let $V$ be a vector space and $S = \{ \vec{s}, \vec{s}_2, \ldots \}$ be a subset of $V$. Then the span of $S$ is the set $\text{span}(S) = \{ c_1 \vec{s}_1 + \cdots + c_n \vec{s}_n \mid c_i \in \mathbb{R}, n \in \mathbb{N} \}$.

**linearly dependent, linearly independent**
The vectors $\vec{v}_1, \ldots, \vec{v}_n$ are linearly dependent if there exists constants $c_1, \ldots, c_n$ not all zero such that $c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n = \vec{0}$. Otherwise, the vectors are linearly independent.

**linear combination**
Any sum of the vectors $\vec{v}_1, \ldots, \vec{v}_n$ multiplied by the scalars $c_1, \ldots, c_n$ namely $c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n$ is a linear combination of $\vec{v}_1, \ldots, \vec{v}_n$.

**basis, finite-dimensional, dimension**
Let $V$ be a vector space. Then $B$ is a basis for $V$ if $B$ is linearly independent and $V = \text{span}(B)$. If $B$ has only a finite number of elements, then we say $V$ is finite-dimensional. The number of vectors in $B$ is the dimension of $V$.

### Theorems/Formulas

**Subspace Theorem**
Let $U$ be a nonempty subset of a vector space $V$. Then $U$ is a subspace of $V$ if and only if $c\vec{u} + d\vec{v} \in U$ for all $\vec{u}, \vec{v} \in U$ and $c, d \in \mathbb{R}$.

For any $S \subseteq V$, $\text{span}(S)$ is a subspace of $V$.

**Linear Dependence**
An ordered set of nonzero vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ is linearly dependent if and only if one of the vectors $\vec{v}_n$ is a linear combination of the preceding vectors.

Let $B = \{ \vec{b}_1, \ldots, \vec{b}_n \}$ be a basis for a vector space $V$. Then every vector $\vec{w} \in V$ can be written uniquely as a linear combination of vectors in the basis $B$: $\vec{w} = c_1 \vec{b}_1 + \cdots + c_n \vec{b}_n$.

If $B = \{ \vec{b}_1, \ldots, \vec{b}_n \}$ is a basis for a vector space $V$ and $T = \{ \vec{w}_1, \ldots, \vec{w}_m \}$ is a linearly independent set of vectors in $V$, then $m \leq n$.

For a finite-dimensional vector space $V$, any two bases for $V$ have the same number of vectors.
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<tr>
<td>domain, codomain</td>
<td>Let ( f: S \rightarrow T ) be a function. Then ( S ) is the domain of ( f ), and ( T ) is the codomain of ( f ).</td>
</tr>
<tr>
<td>range</td>
<td>The range of a function ( f: S \rightarrow T ) is the set ( \text{ran}(f) = { f(s) \mid s \in S } \subseteq T ).</td>
</tr>
<tr>
<td>image</td>
<td>For any subset ( U ) of the domain ( S ) of a function ( f: S \rightarrow T ), the image of ( U ) is ( f(U) = \text{Im}(U) = { f(x) \mid x \in U } ).</td>
</tr>
<tr>
<td>pre-image</td>
<td>The pre-image of any subset ( U \subseteq T ) is ( f^{-1}(U) = { s \in S \mid f(s) \in U } \subseteq S ).</td>
</tr>
<tr>
<td>one-to-one/injective</td>
<td>The function ( f: S \rightarrow T ) is one-to-one (or injective) if for any ( x \neq y \in S ), then ( f(x) \neq f(y) ).</td>
</tr>
<tr>
<td>onto/surjective</td>
<td>The function ( f: S \rightarrow T ) is onto (or surjective) if for any ( t \in T ), there is an ( s \in S ) such that ( f(s) = t ).</td>
</tr>
<tr>
<td>bijective</td>
<td>A function ( f ) is bijective if ( f ) is injective and surjective.</td>
</tr>
<tr>
<td>nullspace/kernel</td>
<td>The nullspace (or kernel) of a linear function ( L: V \rightarrow W ) is the set ( \ker L = { \bar{a} \in V \mid L(\bar{a}) = 0 } \subseteq V ).</td>
</tr>
<tr>
<td>column space, row space</td>
<td>The column space of a matrix is the span of its columns. The row space of a matrix is the span of its rows.</td>
</tr>
<tr>
<td>rank, nullity</td>
<td>The rank of a linear transformation ( L ) is the dimension of its range. The nullity of a linear transformation ( L ) is the dimension of the kernel.</td>
</tr>
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<tr>
<td>A function ( f: S \rightarrow T ) has an inverse function ( g: T \rightarrow S ) if and only if ( f ) is bijective.</td>
<td></td>
</tr>
<tr>
<td>If ( L: V \rightarrow W ) is a linear transformation, then ( \ker L ) is a subspace of ( V ).</td>
<td></td>
</tr>
<tr>
<td>If ( L: V \rightarrow W ) is a linear transformation, then ( \text{ran} L ) is a subspace of ( W ).</td>
<td></td>
</tr>
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</table>

| Dimension Formula               | Let \( L: V \rightarrow W \) be a linear transformation, with \( V \) a finite-dimensional vector space. Then: |
|                                | \[ \dim V = \dim \ker L + \dim \text{ran} L \] |
|                                | \[ = \text{null} L + \text{rank} L \] |
Invertibility

Let $V$ be an $n$-dimensional vector space and suppose $L: V \to V$ is a linear transformation with matrix $M$ is an $n \times n$ matrix, and the following statements are equivalent:

1. The matrix $M$ is invertible.
2. The transpose matrix $M^T$ is invertible.
3. The matrix $M$ is row-equivalent to the identity matrix.
4. If $v \in \mathbb{R}^n$, then $Mx = v$ has exactly one solution.
5. If $v \in V$, then $L(x) = v$ has exactly one solution.
6. The homogeneous solution $Mx = 0$ has no nonzero solutions.
7. The determinant of $M$ is not equal to 0.
8. The columns (or rows) of $M$ span $\mathbb{R}^n$.
9. The columns (or rows) of $M$ are linearly independent.
10. The columns (or rows) of $M$ are a basis for $\mathbb{R}^n$.
11. The linear transformation $L$ is injective.
12. The linear transformation $L$ is surjective.
13. The linear transformation $L$ is bijective.