Exercises:

**Exercise 15:** (Sub-Subsequence criterion) Let $x \in \mathbb{R}$ and $(x_n)$ be a sequence satisfying that every subsequence $(x_{n_k})$ has a subsequence $(x_{n_{k_l}})$ that converges to $x$. Show that $(x_n)$ converges to $x$.

**Exercise 16:** Find an example of the following or explain why it is not possible.

a) Two series $\sum a_n$ and $\sum b_n$ which diverge but $\sum a_n b_n$ converges.

b) A convergent series $\sum a_n$ and a bounded sequence $(b_n)$ such that $\sum a_n b_n$ is divergent.

c) Two sequences $(a_n)$ and $(b_n)$ where $\sum a_n$ and $\sum (a_n + b_n)$ converge but $\sum b_n$ diverges.

d) A sequence $(a_n)$ satisfying $0 \leq a_n \leq \frac{1}{n}$ where $\sum (-1)^n a_n$ diverges.

**Exercise 17:** Let $(a_n)$ be a sequence.

a) Assume $a_n > 0$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} na_n = l \neq 0$. Show that $\sum_{n=1}^{\infty} a_n$ diverges.

b) Assume $a_n > 0$ for all $N \in \mathbb{N}$ and $\lim_{n \to \infty} n^2 a_n = l \neq 0$. Show that $\sum_{n=1}^{\infty} a_n$ converges.

**Exercise 18:** (Dirichlet’s test) Let $(a_n)_{n \in \mathbb{N}_0}$ and $(b_n)_{n \in \mathbb{N}_0}$ be two sequences.

a) Show that

$$\sum_{n=0}^{N} a_n b_n = A_N b_{N+1} + \sum_{n=0}^{N} A_n (b_n - b_{n+1}),$$

where $A_n := \sum_{k=0}^{n} a_k$. This is called summation-by-parts.

b) Moreover, assume that $0 \leq b_{n+1} \leq b_n$ and $\left| \sum_{n=0}^{N} a_n \right| \leq M$ for some $M > 0$ and for all $N \in \mathbb{N}$. Show that

$$\left| \sum_{n=0}^{N} a_n b_n \right| \leq M b_0.$$  \hspace{1cm} (1)
Assume additionally that \( \lim_{n \to \infty} b_n = 0 \) and deduce from that and (1) the convergence of the series

\[
\sum_{n=0}^{\infty} a_n b_n.
\]

Hint: Check Cauchy condition.

Due: Monday October 28 in the lecture.