Exercise 16: Let $c_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} c_n$ be divergent. Let $a_n \geq c_n$. Show that $\sum_{n=1}^{\infty} a_n$ is divergent.

Exercise 17: Check if the following series are convergent, absolutely convergent or divergent.

a) $\sum_{n=1}^{\infty} \frac{n^2}{b_n}$.

b) $\sum_{n=1}^{\infty} (-1)^n \left( \frac{n}{n+1} \right)^n$.

c) $\sum_{n=1}^{\infty} (-1)^n \frac{\cos(n\pi)}{n}$.

d) Let $d(n)$ be the number of decimals of $n$, e.g. $d(4) = 1$, $d(18) = 2$ or $d(301) = 3$. Define $\sum_{n=1}^{\infty} \frac{1}{nd(n)^2}$.

Exercise 18:

a) Let $(a_n)$ be a sequence that converges to 0 and $\sum_{n=1}^{\infty} b_n$ is convergent. Prove or disprove that $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

b) Assume $\sum_{n=1}^{\infty} b_n$ is convergent and $(a_n)$ is an increasing and bounded sequence. Show that $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

Hint: By monotone convergence theorem $(a_n)$ is convergent. You may use Dirichlet’s test.
Exercise 19: Let \((a_n)\) be a sequence such that the sequence \(S_{2n} = \sum_{k=1}^{2n} a_n, n \in \mathbb{N}\) converges. Prove or disprove that \(\sum_{n=1}^{\infty} a_n\) converges.

Exercise 20: Let \(A \subset \mathbb{R}\).

a) Does \(A\) have the same interior as \(\bar{A}\).

b) Prove that the complement of the interior of \(A\) is the closure of \(A^c\).

Exercise 21: Let \((x_n)\) be a sequence with \(x_n \neq x_m\) for all \(n \neq m\). Show that the set of limit points of \((x_n)\) coincides with the set of accumulation points of the set \(\{x_n : n \in \mathbb{N}\}\).

Exercise 22: Let \(A \subset \mathbb{R}\) with \(A \neq \mathbb{R}, \emptyset\). Show that

a) \(\partial A \neq \emptyset\).

b) \(\partial A\) is closed. Hint: You may use here that a set is closed if it contains all limit points.

c) \(\partial A \cup \text{int}(A) = \bar{A}\). Here \(\text{int}(A)\) stands for the interior of a set.

d) \(\bar{A} = A \cup S\), where \(S = \{a \in \mathbb{R} : a\) is accumulation point of \(A\}\).

Exercise 23: Show that the closure of \(\mathbb{Q}\) is \(\mathbb{R}\).