MAT 127A

Name: ____________________________

Student ID.: ____________________________

Have your student ID available on the table.

Please check that you have received four problems.

Solve each problem on the sheet that is provided for it.

Write your last name, first name and student ID on each sheet.

All answers and solutions must provide sufficiently detailed arguments. Write the solutions in the space provided for it right after the exercise.

You have 50 minutes time for your solutions.

Good luck!
Problem 1.  

Let \((x_n) \subset \mathbb{R}\) be a convergent sequence.

a) Provide the definition of \((x_n)\) being convergent.

Solution

\((x_n)\) converges if and only if there exists \(x \in \mathbb{R}\) such that for all \(\epsilon > 0\) there exists \(N \in \mathbb{N}\) such that \(|x_n - x| < \epsilon\) for all \(n > N\).

b) Let \(f : \mathbb{R} \to \mathbb{R}\) be function satisfying \(|f(x) - f(y)| \leq |x - y|^\alpha\) for some \(\alpha > 0\). Show that the sequence \((f(x_n)) \subset \mathbb{R}\) is convergent.

Solution

Let \(\epsilon > 0\). Since \((x_n)\) is convergent there exists \(x \in \mathbb{R}\) and \(N \in \mathbb{N}\) such that

\[|x_n - x| \leq \epsilon^\frac{1}{\alpha}\]

for all \(n > N\). Using the assumption \(|f(x) - f(y)| \leq |x - y|^\alpha\) and the latter, we obtain

\[|f(x_n) - f(x)| \leq |x_n - x|^{\alpha} \leq \left(\epsilon^\frac{1}{\alpha}\right)^{\alpha} = \epsilon\]

for all \(n > N\). This gives the assertion.

c) Find a function \(f : \mathbb{R} \to \mathbb{R}\) and a convergent sequence \((x_n)\) such that \((f(x_n))\) does not converge.

Solution

Consider the sequence \(x_n = \frac{(-1)^n}{n}\). Then \((x_n)\) is convergent with \(x_n \to 0\) as \(n \to \infty\).

On the other hand, let \(f : \mathbb{R} \to \{\pm 1\},\)

\[f(x) = \begin{cases} 
-1, & x < 0 \\
1, & x \geq 0 
\end{cases}\]

Then \(f(x_n) = (-1)^n\) and this sequence is not convergent.
Problem 2. \hspace{1cm} (10 points)

Let \((x_n) \subset \mathbb{R}\) be a sequence.

a) Provide the definition of \(\limsup_{n \to \infty} x_n\).

**Solution**

Define 

\[ y_n = \sup \{ x_k : k \geq n \}. \]

This sequence is convergent by the monotone convergence theorem and

\[ \limsup_{n \to \infty} x_n = \lim_{n \to \infty} y_n. \]

b) Assume for all \(x \in \mathbb{R}\) that \(\lim \inf_{n \to \infty} |x_n - x| > 0\). Show \((x_n)\) is divergent.

**Solution**

Assume \((x_n)\) is convergent. Then there exists \(x \in \mathbb{R}\) such that \(|x_n - x| \to 0\) as \(n \to \infty\). Then

\[ 0 < \lim \inf_{n \to \infty} |x_n - x| \leq \lim_{n \to \infty} |x_n - x| = 0. \]

This is a contradiction, i.e \((x_n)\) diverges.
Problem 3. (10 points)

a) Prove the reverse triangle inequality, i.e. show that for all $x, y \in \mathbb{R}$
\[ ||x| - |y|| \leq |x - y|. \]

Solution

The triangle inequality implies for all $x, y \in \mathbb{R}$ that
\[ |x| = |x - y + y| \leq |x - y| + |y| \quad \text{and} \quad |y| = |y - x + x| \leq |y - x| + |x| \]
which implies
\[ |x| - |y| \leq |x - y| \quad \text{and} \quad |y| - |x| \leq |x - y|. \]

Since $|x| \geq |y|$ or $|y| \geq |x|$ this implies
\[ ||x| - |y|| \leq |x - y|. \]

b) Let $(x_n) \subset \mathbb{R}$ be a convergent sequence. Prove that $|x_n| \to |x|$ as $n \to \infty$.

Solution

Part a) implies
\[ 0 \leq ||x_n| - |x|| \leq |x_n - x|. \]
Since, by assumption $x_n \to x$ as $n \to \infty$ we obtain $|x_n - x| \to 0$ as $n \to \infty$ and the sandwich theorem provides the assertion.

c) Provide an example of a sequence $(x_n)$ which is not convergent but $|x_n| \to |x|$ as $n \to \infty$.

Solution

The sequence $x_n = (-1)^n$, $n \in \mathbb{N}$, is a counterexample.
Problem 4. Let \((x_n)\) be the recursively defined sequence for \(n \geq 2\)

\[ x_{n+1} = 2 - \frac{1}{x_n} \quad \text{and} \quad x_1 = 2. \]

a) Show that \(1 \leq x_n \leq 2\).

Solution

We prove this by induction. For \(n = 1\) it is true by definition of \(x_1\). Let \(n \geq 2\) and assume \(1 \leq x_n \leq 2\). This implies

\[ 1 = 2 - 1 \leq x_{n+1} = 2 - \frac{1}{x_n} \leq 2 - \frac{1}{2} \leq 2. \]

b) Show that \((x_n)\) is decreasing.

Solution

The condition \(x_{n+1} = 2 - \frac{1}{x_n} \leq x_n\) is equivalent to \((x_n - 1)^2 \geq 0\) under the assumption \(x_n \geq 0\) which follows from a). The latter is always true so \(x_{n+1} \leq x_n\) and the sequence decreases.

*Hint: You can show directly that \(x_{n+1} \leq x_n\).*

c) Show that \((x_n)\) is convergent and find its limit.

Solution

Since \((x_n)\) is monotone and bounded the monotone convergence theorem implies convergence of \((x_n)\). Taking the limit \(n \to \infty\) of the recursion equation gives the fixed point equation for the limit \(x\)

\[ x = 2 - \frac{1}{x}, \]

which is equivalent to \((x - 1)^2 = 0\) and therefore \(x = 1\). Hence, \(\lim_{n \to \infty} x_n = 1\).