Exercise 1: Compute the following limits if existing

a) \( \lim_{x \to 0} \frac{1}{\cos(x)} \).

b) \( \lim_{x \to 0} \frac{\sin(x)^2 + \cos(x)^3}{\cos(x)^2 + \sin(x)^4} \).

c) \( \lim_{x \to 0} \sin(x) \sin(e^{\frac{1}{x}}) \).

d) \( \lim_{x \to 0} \frac{\sqrt{(e^x - 1)^2 + 9} - 3}{(e^x - 1)^2} \).

Exercise 2: Let \( N(t) = \frac{1}{1 + t^2} \) and \( t > 0 \).

a) Show that \( N \) solves the differential equation

\[ N' = -2tN^2. \]  

(1)

b) Let \( K > 0 \). Find the solution \( \tilde{N} \) to the differential equation (1) with initial condition \( \tilde{N}(0) = K \) and compute \( \lim_{t \to \infty} \tilde{N}(t) \) for this solution.

Exercise 3: The distance (in meters) an object falls when dropped is given by the formula

\[ s(t) = \frac{1}{2} gt^2 \]

where \( g = 9.81 \frac{m}{s^2} \) and \( t \) is the time.

a) Find the velocity and acceleration of the object.

b) Assume that the object hits the ground with velocity \( v = 5g \). From which height was the object dropped?

Exercise 4: (Tangent line to an ellipse) Let \( a, b > 0 \). We consider the ellipse given by all \( x, y \in \mathbb{R} \) such that

\[ \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 = 1. \]  

(2)

Let \( y_0 < 0 \) and the pair \( (x_0, y_0) \) be on the ellipse, i.e. satisfying equation (2).
a) Find the tangent line to the ellipse in the point \((x_0, y_0)\) using implicit differentiation.

b) Assume additionally that \(a = b\). Show that in this case the normal line to the ellipse going through \((x_0, y_0)\) passes through the point \((0, 0)\).

Exercise 5: Let
\[
f(x) = \begin{cases} 
\sin(x)^2 \sin \left( \frac{1}{x} \right), & x \neq 0 \\
0, & x = 0 
\end{cases}
\]
a) Show that \(f\) is continuous in \(x = 0\).

b) Show that \(f\) is differentiable in \(x = 0\) and compute its derivative in \(x = 0\).

Exercise 6: Let \(f(x) = \frac{1}{1 + x}, \ x > 0\). Let \(n \in \mathbb{N}_0\) and show by induction that
\[
f^{(n)}(x) = \frac{(-1)^n n!}{(1 + x)^{n+1}},
\]
where \(n! = n(n - 1)(n - 2) \cdots 1\).