Calculus for BioSci

Name: ____________________________

Student ID.: _______________________

Have your student ID available on the table.

Please check that you have received four problems.

Solve each problem on the sheet that is provided for it. You can have scratch paper.

Write your last name, first name and student ID on each sheet.

All answers and solutions must provide sufficiently detailed arguments.

Write the solutions in the space provided for it right after the exercise. Do not write on the back.

Calculators are not allowed.

You have 45 minutes time for your solutions.

Good luck!

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Problem 1.  

a) Compute the limit 

\[
\lim_{{x \to \infty}} \frac{x^3 + 1}{2x^3 + 2x^2} = \lim_{{x \to \infty}} \frac{1 + \frac{1}{x^3}}{2 + \frac{2}{x^2}} = \frac{1}{2}
\]

b) Compute the limit 

\[
\lim_{{x \to 0}} \frac{\sqrt{\cos x - 1}^2 + 4 - 2}{(\cos x - 1)^2} = \lim_{{x \to 0}} \frac{\sqrt{\cos x - 1}^2 + 2}{(\cos x - 1)^2}
\]

\[
= \lim_{{x \to 0}} \frac{(\cos x - 1)^2 + 4 - 4}{(\cos x - 1)^2(\cos x - 1)^2 + 4 + 2}
\]

\[
= \lim_{{x \to 0}} \frac{1}{\sqrt{(\cos x - 1)^2 + 4 + 2}} = \frac{1}{4}
\]
Problem 2. Let

\[ N(t) = \frac{1}{\sqrt{1 + t}}, \quad t \geq 0. \]

a) Show that the function \( N \) given above solves the differential equation (6 points)

\[ 2 \frac{dN}{dt} = -N^3, \quad t \geq 0. \]

We compute

\[
2 \frac{dN}{dt} = 2 \frac{d}{dt} \sqrt{1 + t} = 2 \left( -\frac{1}{2} \left( 1 + t \right)^{-\frac{3}{2}} \right) = -\left( \frac{1}{1 + t} \right)^{3/2} = -\left( \frac{1}{\sqrt{1 + t}} \right)^3 = -N^3.
\]

b) Compute the limit (6 points)

\[
\lim_{t \to \infty} \frac{N'(t)}{N(t)^3}.
\]

From a) we know

\[ 2N' = -N^3 \]

\[
\frac{N'}{N^3} = -\frac{1}{2}
\]

\[
\Rightarrow \lim_{t \to \infty} \frac{N'(t)}{N(t)^3} = \lim_{t \to \infty} -\frac{1}{2} = -\frac{1}{2}
\]
Problem 3. We consider the curve given by all points $x, y \in \mathbb{R}$ such that

$$\frac{x^2}{4} + y^3 = 2y.$$

Find the equation of the tangent line through the point $(2, 1) = (x_0, y_0)$ (12 points)

To find the slope of the tangent line, we take $\frac{dy}{dx}$

$$\frac{d}{dx} \left( \frac{x^2}{4} + y^3 \right) = \frac{d}{dx} 2y$$

$$\iff \ 2 \frac{x}{4} + 3y^2 \frac{dy}{dx} = 2 \frac{dy}{dx}$$

$$\iff \ 2 \frac{x}{4} = \frac{dy}{dx} \left( 2 - 3y^2 \right)$$

$$\iff \ \frac{dy}{dx} = \frac{2x}{4(2-3y^2)} \quad \Rightarrow \quad \text{insert} \ (x_0, y_0) = (2, 1)$$

$$\frac{dy}{dx}(2) = \frac{4}{4(2-3)} = -1$$

$$\Rightarrow \ z = -x + c$$

\text{insert} (2, 1) \quad \Rightarrow \ 2 = -1 + c \quad \iff \ c = 3$$

$$\Rightarrow \ z = -x + 3 \quad \text{is the equation for the tangent line.}$$
Problem 4. Let

\[ f(x) = \begin{cases} 
  x^2 \cos \left( \frac{1}{x} \right) + x, & x \neq 0 \\
  0, & x = 0 
\end{cases} \]

a) Show that \( f \) is continuous at \( x = 0 \).

\[
\lim_{x \to 0} \left( x^2 \cos \left( \frac{1}{x} \right) + x \right) = \lim_{x \to 0} x^2 \cos \left( \frac{1}{x} \right) + \lim_{x \to 0} x \\
= 0 + 0 = 0
\]

Now \( -x^2 \leq x^2 \cos \frac{1}{x} \leq x^2 \). Since \( \lim_{x \to 0} x^2 = 0 \), the sandwich theorem implies

\[
\lim_{x \to 0} x^2 \cos \left( \frac{1}{x} \right) = 0 \quad \text{and with (x) this gives}
\]

\[
\lim_{x \to 0} f(x) = f(0) = 0 \quad \text{and \( f \) is continuous in} \ 0.\]

b) Show that \( f \) is differentiable at \( x = 0 \) and compute its derivative at \( x = 0 \).

\[
\lim_{h \to 0} \frac{h^2 \cos \left( \frac{1}{h} \right) + h}{h} = \lim_{h \to 0} h \cos \left( \frac{1}{h} \right) + 1 \quad (x)
\]

Using the sandwich theorem as in part a) we conclude

\[ \lim_{h \to 0} h \cos \left( \frac{1}{h} \right) = 0 \]

and therefore \( f'(0) = 1 \)

and \( f \) is diff. with \( f'(0) = 1 \).