Lieb-Robinson bounds for a class of continuum many-body fermion systems

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joint work with B. Nachtergaele, J. Reschke and R. Sims
Consider the Schrödinger equation on $\ell^2(\mathbb{Z}^d)$

$$-i\frac{d}{dt}\varphi_t = (-\Delta_d + V)\varphi_t$$

$$\varphi_0 = \delta_0$$

Solution is $\varphi_t = e^{-itH}\delta_0$ with $H = -\Delta_d + V$. Find estimates on

$$|\langle e^{-itH}\delta_0 \rangle(n)| \leq ???$$

Trotter product formula and decay of $\langle \delta_k, e^{-it(-\Delta_d)}\delta_m \rangle$ imply

$$|\langle e^{-itH}\delta_0 \rangle(n)| = |\langle \delta_n, e^{-itH}\delta_0 \rangle| \leq Ce^{c_1 t} e^{-c_2 n}$$

independently of $V \in \ell^\infty(\mathbb{Z}^d)$. 

If you sit at distance $n$ it takes (at least) order $t \sim n$ until you feel something (ballistic propagation). Finite speed of propagation in non-relativistic model.

We are interested in analogue bounds for many-body systems.
What are Lieb-Robinson bounds?

- Spin-chain, i.e. \( \mathcal{H}_N = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \) and Hamiltonian

  \[
  H^{(N)} = H_{12} + H_{23} + \cdots + H_{N-1N} \quad \text{(short-range)}
  \]

- \( A, B \in BL(\mathbb{C}^2) \)

  \[
  \delta_0 \equiv \text{id} \otimes \cdots \otimes A \otimes \cdots \otimes \text{id} \otimes \cdots \otimes \text{id}
  \]
  \[
  \delta_n \equiv \text{id} \otimes \cdots \otimes \text{id} \otimes \cdots \otimes B \otimes \cdots \otimes \text{id}
  \]

- Time evolution of \( A \)

  \[
  e^{itH} A e^{-itH} \quad \text{(Heisenberg evolution)}
  \]

  and one can estimate

  \[
  \left\| [e^{itH} A e^{-itH}, B] \right\| \leq C e^{c(vt - \text{dist}(A, B))} \quad \text{(LR-bound)}
  \]

  Until \( t \sim \text{dist}(A, B) \) commutator is exp. small. Propagation is at most ballistic.

- Lieb-Robinson 60ies: Finite speed of propagation in non-relativistic model!
Consider interacting Fermions in representation on Fock space.

- For fixed particle number $N \in \mathbb{N}$

$$H_N = \sum_{k=1}^N \left( -\Delta_k + V(x_k) \right) + \sum_{1 \leq k < l \leq N} W(x_k - x_l)$$

acting on $(L^2(\mathbb{R}^N \times \mathbb{R}^d))^-$ (anti-symmetric) with $V, W \in L^\infty(\mathbb{R}^d)$.

- We don’t want to consider fixed particle number! Introduce

$$\mathcal{F}^- = \bigoplus_{N=0}^\infty (L^2(\mathbb{R}^N \times \mathbb{R}^d))^-(\text{anti-symm. Fock space})$$

Then

$$\bigoplus_{N=0}^\infty H_N = \bigoplus_{N=0}^\infty \left( \sum_{k=1}^N \left( -\Delta_k + V(x_k) \right) + \sum_{1 \leq k < l \leq N} W(x_k - x_l) \right)$$

$$= \int_{\mathbb{R}^d} dx \ a_x^* (-\Delta + V) a_x + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dx dy \ W(x - y) a_x^* a_y^* a_y a_x$$
Model: Interacting Fermions in $\mathbb{R}^d$

\[ \mathcal{H} = \int_{\mathbb{R}^d} dx \ a_x^* (-\Delta + V) a_x + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dx dy \ W(x - y) a_x^* a_y^* a_y a_x \]

- $a_x = a(\delta_x)$ and $a_x^* = (a(\delta_x))^*$ are annihilation and creation operators of a particle in state $\delta_x$.

- More generally $a(f)$, $a(g)$ satisfy CAR

\[ \{a(f), a(g)\} = 0 \quad \text{and} \quad \{a(f), a^*(g)\} = \langle f, g \rangle \mathbb{1} \]

For all $f, g \in L^2(\mathbb{R}^d)$

\[ \|a(f)\| = \|a^*(f)\| = \|f\|_2 \]

- Introduce infra-red and ultra-violet cutoff to make interaction bounded.
\[ H = \int_{\mathbb{R}^d} dx \ a_x^* (-\Delta + V) a_x + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dx dy \ W(x - y) a_x^* a_y^* a_y a_x \]

- Introduce infra-red and ultra-violet cutoff

\[ H^\sigma_{\Lambda_L} = d\Gamma(-\Delta + V) + W^\sigma_{\Lambda} \]

with \( \Lambda_L = [-L, L]^d \), \( \sigma > 0 \)

\[ W^\sigma_{\Lambda} = \frac{1}{2} \int_{\Lambda_L} \int_{\Lambda_L} dx \ dy \ W(x - y) a_x^* (\phi^\sigma_x) a^*(\phi^\sigma_y) a(\phi^\sigma_y) a(\phi^\sigma_x) \]

and Gaussian

\[ \phi^\sigma_x(y) = \frac{1}{(2\pi\sigma^2)^{d/2}} e^{-\frac{|y - x|^2}{2\sigma^2}} \]

- Particles interact as having finite size \( \sigma \)
- Particles only interact in box \( \Lambda_L \).
\[ H^{\sigma}_{\Lambda_L} = d\Gamma(-\Delta + V) + \frac{1}{2} \int_{\Lambda_L} \int_{\Lambda_L} dx \, dy \, W(x - y) a^*(\varphi^\sigma_x) a^*(\varphi^\sigma_y) a(\varphi^\sigma_y) a(\varphi^\sigma_x) \]

- \( \sigma > 0 \) is ‘size’ of particles
- For \( \sigma \to 0 \), \( \varphi^\sigma_x \to \delta_x \). For \( \sigma \) small our model should be close to initial model

**Consistency:**

**Lemma.**

For real-valued \( V, W \in L^\infty(\mathbb{R}^d) \) and compact \( \Lambda \subset \mathbb{R}^d \) (fix infra-red cutoff)

\[ H^{\sigma}_{\Lambda} \to H_{\Lambda} \]

in the strong resolvent sense as \( \sigma \downarrow 0 \).

- Fix particle size \( \sigma > 0 \) and our goal is to take \( \Lambda \to \infty \) in terms of dynamics
Background potential and interaction

\[ H_{\Lambda_L}^\sigma = d\Gamma(-\Delta + V) + \frac{1}{2} \int_{\Lambda_L} \int_{\Lambda_L} dx \, dy \, W(x - y) a^*(\varphi_x^\sigma) a^*(\varphi_y^\sigma) a(\varphi_y^\sigma) a(\varphi_x^\sigma) \]

- Interaction: \( W \) symmetric and
  \[ |W(x)| \leq Ce^{-c|x|} \] (short-range interaction)

- Background potential \( V \): Fourier transform of signed, compactly supported, finite measure \( \mu \), which is real-valued and bounded, i.e.
  \[ V(x) = \int_{B_M} d\mu(k) e^{-ik \cdot x} \]

Important: Finite support in momentum space!
- \( V(x) = \cos(x) \) (any real-valued trigonometric polynomial works)
- \( V(x) = \text{sinc}^k(x) \) (\( d\mu(x) = 1_{(-1,1)} dx \))

- We wanted to include periodic background potential \( V \).
  Spectral gap persists for small coupling?
Model: The CAR algebra

- For $A \in \mathcal{B}(\mathcal{F}^-)$ and $t \in \mathbb{R}$

$$\tau_t^\Lambda(A) = e^{itH^\Lambda} Ae^{-itH^\Lambda}$$

(Heisenberg dynamics).

We analyze this dynamics not for all $A \in \mathcal{B}(\mathcal{F}^-)$ only for

$$A \in \overline{\mathcal{A}(\{a(f), a^*(f) : f \in L^2(\mathbb{R}^d)\})}\|\cdot\|$$

(CAR-Algebra)

- $f, g \in L^2(\mathbb{R}^d)$ with supp $f \cap$ supp $g = \emptyset$ then by the CAR relations

$$\{a(f), a(g)\} = 0 \quad \text{and} \quad \{a(f), a^*(g)\} = \langle f, g \rangle 1 = 0$$

\{·, ·\} is anti-commutator. We are interested in bounds on

$$\{\tau_t^\Lambda(a(f)), a(g)\} \quad \text{and} \quad \{\tau_t^\Lambda(a(f)), a^*(g)\}$$
We aim at estimating for \( f, g \in L^2(\mathbb{R}^d) \) with \( \text{supp} f \cap \text{supp} g = \emptyset \)

\[
\left\| \{ \tau_t^\Lambda (a(f)), a(g) \} \right\| + \left\| \{ \tau_t^\Lambda (a(f)), a^*(g) \} \right\|
\]

but there is a caviat!

For non-interacting system \( W = 0 \)

\[\tau_t^0(a(f)) = a(e^{-it(-\Delta+V)}f)\]

and therefore

\[
\left\| \{ \tau_t^0(a(f)), a^*(g) \} \right\| = \left\| \{ a(e^{-it(-\Delta+V)}f), a^*(g) \} \right\| = |\langle e^{-it(-\Delta+V)}f, g \rangle|
\]

This does for general \( f, g \in L^2_c(\mathbb{R}^d) \) not decay exponentially in \( \text{dist}(\text{supp} f, \text{supp} g) \), e.g. for \( V = 0 \)

\[e^{-it(-\Delta)}(x, y) = \frac{1}{(4\pi it)^{d/2}} e^{\frac{i|x-y|^2}{4t}} \quad (\text{non-relativistic operator})\]

We aim at estimating difference to free \((W = 0)\) dynamics

\[F_t^\Lambda(f, g) = \left\| \{ \tau_t^\Lambda (a(f)), a^*(g) \} - \{ \tau_t^0(a(f)), a^*(g) \} \right\| + \left\| \{ \tau_t^\Lambda (a(f)), a(g) \} \right\|\]
Lieb-Robinson bounds for interacting Fermions

\[ F^\Lambda_t(f, g) = \| \{ \tau^\Lambda_t(a(f)), a^*(g) \} - \{ \tau^\emptyset_t(a(f)), a^*(g) \} \| + \| \{ \tau^\Lambda_t(a(f)), a(g) \} \| \]

**Theorem** (Many-body Lieb-Robinson bound GNRS ’20).

Fix \( \sigma > 0 \) and \( W \) exponentially decaying and \( V \) with finite support in momentum space. Then there exist constants \( C_1, C_2 > 0 \) such that for all compact \( \Lambda \subset \mathbb{R}^d \), \( t \in \mathbb{R} \) and any \( f, g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \)

\[ F^\Lambda_t(f, g) \leq \| f \|_1 \| g \|_1 e^{C_1 t^d + \frac{d(f, g)}{1 + t^2}} e^{-C_2 \frac{d(f, g)}{(1 + t^2)}} \]

where \( d(f, g) = \text{dist}(\text{supp } f, \text{supp } g) \).

- Decay as long as \( t^{6d+3} < d(f, g) \). The exponent \( 6d + 3 \) is most likely not optimal.
- Finite speed of propagation.
- Independent of \( \Lambda \) but constants explode for \( \sigma \to 0 \).
Application: Thermodynamic limit of dynamics exists

*Theorem* (Many-body Lieb-Robinson bound GNRS ’20).
There exists a strongly continuous one-parameter group of automorphisms of the CAR algebra over $L^2(\mathbb{R}^d)$, $\{\tau_t\}_{t \in \mathbb{R}}$, such that for all $f \in L^2(\mathbb{R}^d)$ and any increasing sequence $(\Lambda_n)$ of compact sets such that $\bigcup_n \Lambda_n = \mathbb{R}^d$,

$$\lim_{n \to \infty} \tau_t^{\Lambda_n}(a(f)) = \tau_t(a(f))$$

in the operator norm topology, with convergence uniform in $t$ on compact subsets of $\mathbb{R}$.

- Infinite volume dynamics exists.
- Independent of representation.
- The theorem implies strong continuity of $t \mapsto \tau_t(\cdot)$. This is important as by Stone’s theorem we get generator (in GNS-representation): 'Infinite-volume operator'.
**Idea proof: Lieb-Robinson bounds**

\( F^\Lambda_t(f, g) = \|\{\tau^\Lambda_t(a(f)), a^*(g)\} - \{\tau^\emptyset_t(a(f)), a^*(g)\}\| + \|\{\tau^\Lambda_t(a(f)), a(g)\}\| \)

- **Interaction picture implies**

\[
\tau^\Lambda_t(a(f)) = \tau^\emptyset_t(a(f)) + i \int_0^t ds \tau^\Lambda_s \left( [W^\sigma, \tau^\emptyset_{t-s}(a(f))] \right)
\]

- **\( F^\Lambda_t(f, g) \leq \sum_{n \in \mathbb{N}} a_n(t, f, g) \)** with

\[
a_n(t, f, g) = C^n_\sigma \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \int_{\mathbb{R}^d} dx_1 \cdots \int_{\mathbb{R}^d} dx_n K_{t-t_1}(f, x_1) \times K_{t_1-t_2}(\varphi_{x_1}^\sigma, x_2) \cdots K_{t_{n-1}-t_n}(\varphi_{x_{n-1}}^\sigma, x_n) |\langle e^{-it_1 H_1} \varphi_{x_1}^\sigma, g \rangle|
\]

and \( K_t(f, x) = \|W\|_1 |\langle e^{-itH_1} f, \varphi_x^\sigma \rangle| + 2 \left( |W| * |\langle e^{-itH_1} f, \varphi_x^\sigma \rangle| \right)(x) \)

- **For our proof it is essential to bound**

\[
|\langle e^{-itH_1} \varphi_y^\sigma, g \rangle| \leq ???
\]

for one-particle operator \( H_1 = -\Delta + V \). Gaussian important here.
One-particle Lieb-Robinson bounds on $\mathbb{R}^d$

- Estimate

$$\left| \left( e^{-it(-\Delta+V)} \varphi_{y}^{\sigma} \right)(x) \right| \leq ??$$

- For $V = 0$:

$$\left| \left( e^{-it(-\Delta)} \varphi_{y}^{1} \right)(x) \right| = \frac{1}{(2\pi)^{d/2}} \frac{e^{-\frac{|x-y|^2}{8t^2+2}}}{(4t^2+1)^{d/4}}$$

**Theorem** (One-particle Lieb-Robinson bounds, GNRS '20).

Let $V$ be compactly supported in momentum space, i.e. $V(x) = \int_{B_M} d\mu(k) \, e^{-ik \cdot x}$. Then there exist constants $C_1, C_2, C_3 > 0$, such that for all $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$

$$\left| e^{-it(-\Delta+V)} \varphi_{y}^{\sigma} (x) \right| \leq C_1 e^{C_2 |t| \ln |t|} e^{-C_3 \frac{|x-y|}{1+t^2}}$$
**Theorem** (One-particle Lieb-Robinson bounds, GNRS ’20).

Let $V$ be compactly supported in momentum space, i.e. $V(x) = \int_{B_M} d\mu(k) e^{-ik \cdot x}$. Then there exist constants $C_1, C_2, C_3 > 0$, such that for all $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$

$$|e^{-it(\Delta + V)} \varphi_x^\sigma| \leq C_1 e^{C_2 |t| \ln |t|} e^{-C_3 \frac{|x-y|}{1+t^2}}$$

- No quadratic decay.
- In the discrete case independently of $V \in l^\infty(\mathbb{Z}^d)$

$$|(e^{-it(-\Delta_0 + V)} \delta_n)(m)| \leq e^{D_1 t - D_2 |n-m|}$$

Exercise in [AW17].

- Bound independent of $V \in L^\infty(\mathbb{R}^d)$? Does bound hold for all $V \in L^\infty(\mathbb{R}^d)$? Other interesting cases are harmonic oscillator or constant magnetic field. Remember we want spectral gap.
Idea proof: One-particle Lieb–Robinson bounds

Let \( H_0 = -\Delta \) and \( H_1 = -\Delta + V \).

- Use Dyson series

\[
e^{-itH_1} = e^{-itH_0} + (-i)^n \sum_{n=1}^{\infty} \int_0^t dt_n \cdots \int_0^t dt_1 \\
\times e^{-i(t-t_n)H_0}Ve^{-i(t_n-t_{n-1})H_0}V \cdots Ve^{it_1H_0}
\]

which follows from Duhamel’s formula

\[
e^{-isH_1}e^{-i(t-s)H_0} \bigg|_{s=0}^{s=t} = i \int_0^t ds e^{-isH_1}Ve^{-i(t-s)H_0}
\]

- Write \( V(x) = \int_{B_M} d\mu(k) e^{-ik \cdot x} \)

\[
\int_{B_M} d\mu(k_1) \cdots \int_{B_M} d\mu(k_n) \int_0^t dt_n \cdots \int_0^t dt_1 \\
\left( e^{-i(t-t_n)H_0} e^{-ik_1 x_1} e^{-i(t_n-t_{n-1})H_0} e^{-ik_2 x_2} \cdots e^{-ik_n x_n} e^{it_1H_0} \varphi_y \right)(x)
\]

- Going to momentum space the latter can be computed explicitly
\[
\frac{e^{-\frac{\sigma^2}{8t^2 + 2\sigma^4}}} \left| (x-y) - 2 \sum_{l=0}^{n} (t_{l+1} - t_{l}) \sum_{j=1}^{l} k_j \right|^2}{(4t^2 + \sigma^4)^{d/4}}
\]

▶ We continue

\[
\frac{e^{-\frac{\sigma^2}{8t^2 + 2\sigma^4}}} \left| (x-y) - 2 \sum_{l=0}^{n} (t_{l+1} - t_{l}) \sum_{j=1}^{l} k_j \right|^2}{(4t^2 + \sigma^4)^{d/4}} \leq C_{t,d} e^{-C(|x-y|-Mnt)^2}
\]

▶ Estimate

\[
\left| \left( \sum_{n=1}^{\infty} \int_{0}^{t} dt_{n} \cdots \int_{0}^{t_2} dt_{1} e^{-i(t-t_n)H_0} V \cdots V e^{it_1H_0} \varphi^\sigma \right) (x) \right|
\]

\[
\leq \sum_{n \in \mathbb{N}} \frac{\mu(\mathbb{R})^n t^n}{n!} C_{t,d} e^{-C(|x-y|-Mnt)^2}
\]

▶ Analyze series and get result for one-particle.
Proof many-body LR-bound:

\[ K_t(\varphi_y, x) = \|W\|_1 |\langle e^{-itH_1} \varphi^\sigma_y, \varphi^\sigma_x \rangle| + 2 \left( |W| \ast |\langle e^{-itH_1} \varphi^\sigma_y, \varphi^\sigma_{(\cdot)} \rangle| \right)(x) \]

- Insert one-particle bound in many-body bound ...

\[ |K_t(\varphi_y, x)| \leq Ce^{t\ln t} e^{-ct|x-y|} \]

- End up with lots of convolution operators ...
- Take Fourier transform ...
- We also need estimates on modified Bessel functions ...
Open problems

\[ H_{\Lambda}^\sigma = d\Gamma(-\Delta + V) + \lambda \int_{\Lambda} \int_{\Lambda} dx \, dy \, W(x - y) a^*(\varphi^\sigma_y) a^*(\varphi^\sigma_y) a(\varphi^\sigma_y) a(\varphi^\sigma_x) \]

Let \( V \) be periodic such that \( \text{spec}(-\Delta + V) \) has gap. Choose Fermi energy in this gap. GNS Hamiltonian of \( d\Gamma(-\Delta + V - E) \) has a gap above its ground-state. Does the gap stay open for \( \lambda \) small independently of \( \Lambda \). Known for interacting fermions on lattice, see e.g. de Roeck-Salmhofer ’17.

Show one particle Lieb-Robinson bounds for arbitrary \( V \in L^\infty(\mathbb{R}^d) \) or \( V \) harmonic oscillator or in a constant magnetic field

\[ \left| \left( e^{-it(-\Delta+V)} \varphi^\sigma_y \right)(x) \right| \leq C_1 e^{\mathcal{P}(t)} e^{-C_2|x-y|} \]

for some polynomial \( \mathcal{P} \).
From such a bound the many-body bound follows rather directly.
Thank you for your attention!

References

[1] M. Gebert, B. Nachtergaele, J. Reschke and R. Sims
Lieb-Robinson bounds and strongly continuous dynamics for a class of many-body fermion systems in $\mathbb{R}^d$. arXiv:1912.12552