You have two hours to complete this exam. No books, references, cell phones, or communication of any sort is allowed during the exam. Write your answers in the space given below each problem, and if you need more room for a problem, you can go to the back of a page.

The table below is not to be filled in by you; it is for grading use only. (Grading scheme coming soon)
1. Three wolves, Alpha, Beta, and Gamma, have peculiar language habits. One of them (the truth-teller) always tells the truth, another (the liar) always lies, and the third (the copycat) tells the truth if and only if the previously spoken statement it heard was true.

They have the following conversation:

   **Alpha**: “Gamma always lies.”
   **Beta**: “I never lie.”
   **Gamma**: “Yes, you do!”
   **Alpha**: “No, he doesn’t!”

Which is the liar, which is the truth teller, and which is the copycat? Explain your answer, and show it is the only valid solution.
2. Computational problems (25 points):

(a) Today is Thursday. What day of the week will it be \(3^{500}\) days from today?

(b) Evaluate this product as a simplified permutation in cycle notation, and then compute its inverse:

\[(1532)(476)(1735)(24) =\]

(c) There are 105 cows in the UC Davis Dairy Teaching and Research Facility. Of them, 50 are spotted, 45 are docile (the others are aggressive), and 46 are overweight. Exactly 15 are both spotted and docile, exactly 20 are both docile and overweight, and exactly 16 are both spotted and overweight. Five of the cows have none of these three properties. How many spotted, aggressive, overweight cows are there in the facility?
3. True or False: (1 point each) State whether each of the following is True or False. (Write out the entire word “True” or “False” for your answer for each, rather than just T or F, so that we can easily read what you wrote.)

(a) \(\{1, 2, 3\} \subseteq \{\{1, 2, 3\}\}\)

(b) \(\{a, c, e\} \subseteq \{a, b, c\} \cup \{c, e, f\}\)

(c) \(N \in \mathcal{P}(N)\)

(d) \(\forall A, \emptyset \subseteq A\)

(e) \(\forall A, A \nsubseteq \emptyset\)

(f) \(\forall A, \exists B, A \cap B = \emptyset\) and \((A \cup B) \subseteq A\)

4. Injective, Surjective, Bijective, oh my! For each of the following functions, determine whether it is injective and whether it is surjective. If it is bijective, find its inverse.

(a) Let \(f : \mathbb{R} \to \mathbb{R}\) by \(f(x) = 2x + 1\). Circle all that apply to \(f\):

- Injective
- Surjective
- Bijective
- (If Bijective) Inverse: _____________________________

(b) Let \(g : \mathbb{Z} \to \mathbb{N}\) by \(g(x) = |x|\). Circle all that apply to \(g\):

- Injective
- Surjective
- Bijective
- (If Bijective) Inverse: _____________________________
5. Is it a Group? Is it a Ring? No... it’s a Field!

(a) For each of the following, determine whether or not it is a group. If it is not a group, state why.

- \((\mathbb{R}, \cdot)\)
- \((\mathbb{Z}, +)\)
- \((\mathbb{Z}/6\mathbb{Z}, \cdot)\)
- \((\mathbb{N}, +)\)
- \((\{\text{id}, (123), (132)\}, \circ)\)

(b) Which of the following are rings? If it is not a ring, state why. If it is a ring, determine whether it is also a field.

- \((S_n, \circ, \circ)\)
- \((\{0\}, +, \cdot)\)
- \((\mathbb{Z}, +, \cdot)\)
- \((\mathbb{C}, +, \cdot)\)
6. Fill in the input line numbers for each of the reasons in the following formal proof in the Predicate Logic axiom system. Blanks (\omega) are given for each of the numbers that should be filled in. Each line is one point.

**Theorem 1.** \((\exists x, P(x)) \Rightarrow (\forall x, P(x) \Rightarrow Q)\)

**Proof.**

1. Assume \((\exists x, P(x)) \Rightarrow Q\) -
2. Let \(y\) be arbitrary -
3. Assume \(P(y)\) -
4. \(\exists x, P(x)\) \(\exists +; \omega\)
5. \(Q\) \(\Rightarrow -; \omega \omega\)
6. \(\leftarrow\) -
7. \(P(y) \Rightarrow Q\) \(\Rightarrow +; \omega \omega \omega\)
8. \(\leftarrow\) -
9. \(\forall x, P(x) \Rightarrow Q\) \(\forall +; \omega \omega \omega\)
10. \(\leftarrow\) -
11. \((\exists x, P(x)) \Rightarrow Q\) \(\Rightarrow \omega; \omega \omega \omega\)
12. Assume \(\forall x, P(x) \Rightarrow Q\) -
13. Assume \(\exists x, P(x)\) -
14. For some \(z\) -
15. \(P(z)\) \(\exists -; \omega\)
16. \(P(z) \Rightarrow Q\) \(\forall -; \omega\)
17. \(Q\) \(\Rightarrow -; \omega \omega\)
18. \(\leftarrow\) -
19. \((\exists x, P(x)) \Rightarrow Q\) \(\Rightarrow +; \omega \omega \omega\)
20. \(\leftarrow\) -
21. \((\forall x, P(x) \Rightarrow Q) \Rightarrow (\exists x, P(x) \Rightarrow Q)\) \(\Rightarrow +; \omega \omega \omega\)
22. \((\exists x, P(x)) \Rightarrow Q\) \(\iff (\forall x, P(x) \Rightarrow Q)\) \(\iff +; \omega \omega\)
7. (20 points) Define the sequence $b_0, b_1, b_2, \ldots$ by the recursion $b_0 = 2$, $b_1 = 5$, and $b_n = 5b_{n-1} - 6b_{n-2}$ for all $n \geq 2$. Use strong induction to prove that for all $n \geq 0$, $b_n = 2^n + 3^n$. 
8. (20 points) Prove that, for any sets $A, B, C$, if $A \subseteq B$ then $C - B \subseteq C - A$. Is the converse of this statement true? Why or why not?

(You must be fully rigorous in your use of definitions for subsets and set operations.)
9. Give a combinatorial proof of the identity

\[
\binom{n}{m} \binom{n-m}{k} = \binom{n}{k} \binom{n-k}{m}.
\]

(You may NOT use algebra, induction, factorials, or any other method. The proof must be entirely combinatorial.)
10. Prove that the set $\mathbb{N} \times \mathbb{N}$ is countable.
11. Let $G$ be a group and let $a \in G$. The **centralizer** of $a$ in $G$, written $C_G(a)$, is the set

$$C_G(a) = \{x \in G : xa = ax\}.$$  

In other words, it is the set of all elements that commute with $a$ under the group operation. Prove that $C_G(a)$ is a subgroup of $G$. 