Math 108 - Introduction to Abstract Mathematics

Project Prompts

1 Instructions

Choose one of the following four prompts to think and write about. These open-ended problems are designed to give you practice thinking about a difficult mathematical problem for a longer period of time, and to give you practice with expository mathematical writing beyond just a single problem solution. Write at least four typed pages (in LaTeX) on what you have discovered in your investigations. The writing should consist of the following components:

- Write a brief introduction/exposition to explain what the problem is and what you will be proving or conjecturing.
- The mathematical content can be either solutions to some of the suggested questions below each prompt, or relevant facts that you have discovered on your own - or a mix of both.
- Use Theorem/Proof, Definition, Conjecture, and/or Example environments in the LaTeX code, in which most of your mathematical content should be written.
- Write connecting exposition between these LaTeX environments, such as “Now that we have proven the theorem above and have defined our new terminology, we have the tools to prove the following.” Include examples and/or diagrams that make it easier for the reader to follow your work.
- Write a conclusion section that summarizes what you have found and what you think the next natural steps for investigation might be. This section can include things you tried that didn’t work, and how you might get around these problems.
- Adhere to the Tips for Proof Writing on page 9 of the Supplementary Lecture Notes. In general, treat the writing as if you are writing a section of a math textbook for someone unfamiliar with the problem to read and understand.

Due dates and Grading: The first draft of your project is due on the Wednesday before Thanksgiving break; an on-time submission itself, as long as it is at least four pages’ worth of mathematics, will automatically give you 30 out of 100 points towards your final grade on the project. The instructor and TA’s will provide detailed feedback on your draft, and you should then revise your draft according to the feedback.

To submit your first draft:

1. Click on the Share button of your Overleaf file (create an Overleaf file for your project if you didn’t do it in Overleaf) and copy the Read and Edit link.
2. Email the Read and Edit link to me at mgillespie@math.ucdavis.edu. The title of this email MUST be of the form "Project - [Student Name] - [Student ID Number] - [Section time]". Attach a PDF copy of your work to the email, and then send it.
3. You will received typed feedback on your Overleaf documents on a first-come, first-served basis; I will grade in order based on the email timestamp of your submission. So earlier submissions will receive their feedback first and get a head start on editing for the final draft.

Your final draft is due the night before the last day of class. The final draft (which is worth the remaining 70 points) will be graded based on correctness of the proofs, style of the writeup (especially pertaining to suggestions given on your first draft), and relevance of the content to the problem at hand.

This is an independent project. You may not consult other students or any material on the Internet for help with your project. You can ask questions pertaining to the project to your TA or the instructor during office hours.

In summary: Choose your own mathematical adventure, and have fun!
Prompts

1. **Soldiers on a field:** Suppose there are \( n \cdot m \) soldiers standing in an \( n \times m \) grid formation on a field, aligned with east-west and north-south directions. The commander, who is standing north of the grid, shouts “Attention!”. However, it is such a windy day that some of the soldiers mistook his command for “Right face!” or “Left face!” or “About face!” instead. In their confusion, some of the soldiers jump to face north, some east, others west, and the rest south.

Each second after that, if a soldier \( A \) is facing an adjacent soldier \( B \) that is not facing the same direction, then \( A \) thinks they have made an error and jumps to face \( B \)'s current direction. All such soldiers jump simultaneously each second. What types of patterns can form in their jumping sequences?

Here are some questions you can ponder to get you started on your investigation:

(a) Consider the case in which \( n = m = 2 \), so there is a \( 2 \times 2 \) grid of four soldiers. Suppose they start in the following configuration:

\[
\rightarrow \downarrow \\
\uparrow \leftarrow
\]

List out the next few configurations as the soldiers jump. What pattern do you notice?

(b) What cyclic patterns of soldiers can you find, in other words, patterns that eventually come back to the starting position and repeat? What are the lengths of these cycles? Can you find cycles with any integer length? Can you find an infinite non-repeating pattern of soldiers?

(c) A fixed point in a system such as this is a configuration that doesn’t change, in other words, a starting configuration of soldiers from which no soldier ever changes direction. Can you come up with an explicit description of all the fixed points in this setting? What kinds of starting configurations eventually stop changing?

(d) Consider the case of a \( 1 \times n \) grid, where all the soldiers are in a row. Can you fully describe what happens in this case?

2. **The 3x + 1 Problem**, also known as the Collatz Conjecture, is the following famous open (unsolved) problem in mathematics. Let \( C : \mathbb{Z} \rightarrow \mathbb{Z} \) be the function defined by

\[
C(x) = \begin{cases} 
3x + 1 & \text{if } x \text{ is odd} \\
\frac{x}{2} & \text{if } x \text{ is even}
\end{cases}
\]

For instance, \( C(3) = 3 \cdot 3 + 1 = 10 \) since 3 is odd, and \( C(10) = 5 \) since 10 is even. The conjecture states that if we repeatedly apply the function \( C \) to any positive integer, eventually we reach the number 1. For instance, applying \( C \) repeatedly starting at 3, we get the sequence:

\[3, 10, 5, 16, 8, 4, 2, 1, \ldots\]

which does indeed reach the number 1 after seven steps. What can you say about this problem? (Warning: If you can prove the entire conjecture - that you always reach 1 no matter which positive integer you start with - you will be instantly famous!)

Here are some possible places to start your investigation:

(a) What seems to happen if we start with a negative integer instead? What happens when we start with 0? What happens if we continue the sequence above after reaching 1?

(b) Show that any number of the form \( 2^n \) eventually goes to 1 under repeated applications of \( C \).

(c) Show that any number of the form \( 2^n \cdot 3 + 1 \) also eventually goes to 1, as long as that number is an integer. For which values of \( n \) is \( \frac{2^n - 1}{3} \) an integer?

(d) Can you find other infinite families of numbers that you can prove eventually go to 1?
(e) What happens if you change the function to

\[ A(x) = \begin{cases} 
  x + 1 & \text{if } x \text{ is odd} \\
  x/2 & \text{if } x \text{ is even}
\end{cases} \]

or to

\[ B(x) = \begin{cases} 
  x + 5 & \text{if } x \text{ is odd} \\
  x/2 & \text{if } x \text{ is even}
\end{cases} \]

or another similar function?

3. Stair Climbing Olympics: (If you choose this prompt, you must solve part (a) as a lemma in your writeup, and then go on to investigate what you can in the next parts.) In the Stair Climbing Olympics, each contestant must present their own stair climbing dance from the bottom of the stairs to the top. How might different climbing rules affect the number of possible stair climbing dances?

(a) To warm up, participants jog from the bottom of the stairs to the top. They may climb stairs one at a time or two at a time. For instance, if they are climbing from step 1 to step 9, they might start at stair 1 and take the sequence of steps 1, 3, 4, 5, 7, 8, 9. If they are jogging on a length \( n \) staircase starting on step 1 and ending on step \( n \), let \( W_n \) be the number of ways they can warm up (in terms of \( n \)). Compute \( W_1 \), \( W_2 \), and \( W_3 \), and show that

\[ W_n = W_{n-1} + W_{n-2} \]

for all \( n \geq 3 \). Use this recursion to compute \( W_{10} \).

(b) In Round 1, the contestants must only step up or down by 1 or 2 stairs on each step, and must start at stair 1 and end at stair \( n \), stepping on every stair exactly once. For instance, if \( n = 9 \), they might step on the stairs in the order:

\[ 1, 3, 2, 4, 5, 6, 8, 7, 9. \]

Let \( R_n \) be the number of ways of climbing a staircase of length \( n \) in this manner. Can you find a recursive formula for \( R_n \)?

(c) In Round 2, the contestants may now step up or down by 1, 2, or 3 stairs on each step, and again must start at stair 1 and end at stair \( n \), stepping on every stair exactly once. Let \( F_n \) be the number of ways of climbing a staircase of length \( n \) in this manner. Can you find a recursive formula for \( F_n \)? What kinds of patterns do you notice are possible?

(d) What happens if they can step up or down by any number of steps from 1 up to \( k \) for some \( k \)? Are there other similar and interesting rules you can come up with for a round, for which you can find formulas for the number of possible stair climbs?

4. Singmaster’s conjecture: Recall that Pascal’s Triangle is the infinite triangular array formed by placing 1’s down the sides, and then defining each remaining entry to be the sum of the two numbers just above them:

\[
\begin{array}{cccccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
\]

So for instance, by adding the last row above in pairs, we would get 1, 6, 15, 20, 15, 6, 1 for the next row.
Singmaster’s conjecture is an open problem that states that there is some constant $c$ for which no positive integer besides 1 occurs more than $c$ times in Pascal’s triangle. What can you say about the number of times a given number $m$ might appear in the triangle?

Here are some questions to get you started in your investigation.

(a) The number 1 clearly occurs infinitely many times. Can the number 1 appear anywhere else? Can any other number occur infinitely many times?
(b) Can you find a number besides 1 that appears twice? Four times? Five times? Six times?
(c) The known record is 3003, which occurs eight times. In which rows does it occur?
(d) Suppose two of the entries are equal. Can you use what you know about binomial coefficients and Pascal’s triangle from class to prove some lemmas about some properties they must have?