Problems

Hand in your best work on each of the problems below. Two of the problems in this section will be randomly selected to be graded.

For the first three problems below, give a formal two-column proof of the statement, using the statements and rules of inference of Propositional Logic. Also draw its truth table and verify that it is a tautology.

1. **Theorem:** \([(P \lor Q) \land \neg P) \Rightarrow Q]

   **Proof.**
   
   1. Assume \((P \lor Q) \land \neg P\) -
   2. \(P \lor Q\) and \(-\); 1
   3. \(\neg P\) and \(-\); 1
   4. Assume \(P\) -
   5. Assume \(\neg Q\) -
   6. \(\rightarrow\) \(\rightarrow\) \(+\); 3, 4
   7. \(\leftarrow\) -
   8. \(Q\) \(\leftarrow\); 5, 6, 7
   9. \(\leftarrow\) -
   10. \(P \Rightarrow Q\) \(\Rightarrow\); 4, 8, 9
   11. Assume \(Q\) -
   12. \(\leftarrow\) -
   13. \(Q \Rightarrow Q\) \(\Rightarrow\); 11, 12
   14. \(Q\) or \(-\); 2, 10, 13
   15. \(\leftarrow\) -
   16. \([(P \lor Q) \land \neg P) \Rightarrow Q\] \(\Rightarrow\); 1, 14, 15

\[
\square
\]

2. **Theorem:** \([(P \Rightarrow Q) \land \neg Q) \Rightarrow \neg P]

   **Proof.**
   
   1. Assume \((P \Rightarrow Q) \land \neg Q\) -
   2. \(P \Rightarrow Q\) and \(-\); 1
   3. \(\neg Q\) and \(-\); 1
   4. Assume \(P\) -
   5. \(Q\) \(\Rightarrow\); 2, 4
   6. \(\rightarrow\) \(\rightarrow\) \(+\); 3, 5
   7. \(\leftarrow\) -
   8. \(\neg P\) \(\neg\); 4, 6, 7
   9. \(\leftarrow\) -
   10. \([(P \Rightarrow Q) \land \neg Q) \Rightarrow \neg P\] \(\Rightarrow\); 1, 8, 9

\[
\square
\]
3. **Theorem:** \( \neg(\neg P) \iff P \)

**Proof.**

1. Assume \( \neg(\neg P) \)
2. Assume \( \neg P \)
3. \( \rightarrow \leftrightarrow \) \( \rightarrow \leftrightarrow +; 1, 2 \)
4. \( \leftarrow \)
5. \( P \)
6. \( \leftarrow \)
7. \( \neg(\neg P) \Rightarrow P \)
8. Assume \( P \)
9. Assume \( \neg P \)
10. \( \rightarrow \leftrightarrow +; 8, 9 \)
11. \( \leftarrow \)
12. \( \neg(\neg P) \)
13. \( \leftarrow \)
14. \( P \Rightarrow \neg(\neg P) \)
15. \( \neg(\neg P) \Leftrightarrow P \)

\( \square \)

4. Give an informal proof that \( \sqrt{5} \) is irrational. (You may use any facts you know from basic algebra and arithmetic, and you may use the theorem that if a prime \( p \) divides the product \( a \cdot b \) then either \( p \) divides \( a \) or \( p \) divides \( b \).)

**SOLUTION:** Assume for contradiction that \( \sqrt{5} \) is rational. Then by the definition of rational, \( \sqrt{5} = \frac{a}{b} \), for some integers \( a \) and \( b \) with \( \gcd(a, b) = 1 \). Multiplying both sides by \( b \) and squaring both sides, we have

\[ 5b^2 = a^2. \]

Note that 5 is a prime, and it divides \( 5b^2 \) (by the definition of divides), so 5 divides \( a^2 \) by substitution with the equality above. By Euclid’s lemma, it follows that 5 divides \( a \).

By the definition of divides, there exists a positive integer \( k \) for which \( 5k = a \). Substituting this into the equation above we have

\[ 5b^2 = 25k^2. \]

Dividing both sides we have \( b^2 = 5k^2 \), and so 5 divides \( b^2 \). By similar reasoning to above, it now follows that 5 divides \( b \). But then \( \gcd(a, b) \geq 5 \), a contradiction to the assumption \( \gcd(a, b) = 1 \).

Since we have obtained a contradiction, our original assumption was false and it follows that \( \sqrt{5} \) is in fact irrational. QED

5. Section 1.5, exercise 12 parts (a), (c), (d), (e), (f).

**Extra practice problems**

These are not to be handed in. They are only for your own practice, and are recommended study problems for the midterm or final.

1. Section 1.4, exercises 5, 7, 9.
2. Section 1.5, exercises 4, 6, 7, 9.
3. Prove the following tautologies, each of which show a logical equivalence of two statements:
• $P \lor Q \iff Q \lor P$ (Commutativity of or)
• $P \land Q \iff Q \land P$ (Commutativity of and)
• $(P \lor Q) \lor R \iff P \lor (Q \lor R)$ (Associativity of and)
• $(P \land Q) \land R \iff P \land (Q \land R)$ (Associativity of or)
• $P \lor (Q \land R) \iff (P \lor Q) \land (P \lor R)$ (Distributive Law 1)
• $P \land (Q \lor R) \iff (P \land Q) \lor (P \land R)$ (Distributive Law 2)
• $\neg(P \lor Q) \iff (\neg P \land \neg Q)$ (DeMorgan’s Law 1)
• $\neg(P \land Q) \iff (\neg P \lor \neg Q)$ (DeMorgan’s Law 2)
• $(P \Rightarrow Q) \iff (\neg Q \Rightarrow \neg P)$ (Contrapositive)

### Bonus Problem

Give a formal proof of the statement $P \lor \neg P$.

**Proof.**

1. Assume $\neg(P \lor \neg P)$ -
2. Assume $P$ -
3. $P \lor \neg P$ or $+; 2$
4. $\to\leftarrow$ $\to\leftarrow +; 1, 3$
5. $\leftarrow -$ 
6. $\neg P$ $\neg +; 2, 4, 5$
7. Assume $\neg P$ -
8. $P \lor \neg P$ or $+; 7$
9. $\to\leftarrow$ $\to\leftarrow +; 1, 8$
10. $\leftarrow -$ 
11. $P$ $\neg +; 7, 9, 10$
12. $\to\leftarrow$ $\to\leftarrow +; 6, 11$
13. $\leftarrow -$ 
14. $P \lor \neg P$ $\neg +; 1, 12, 13$