Math 108 - Introduction to Abstract Mathematics
Homework 3 - SOLUTIONS

Problems

Hand in your best work on each of the problems below. Two of the problems in this section will be randomly selected to be graded.

1. Section 1.3, exercise 1 (c), (d), (e), (f), (g)
2. Section 1.3, exercise 8 (a), (c), (g), (h), (l)
3. Section 1.3, exercise 10 (a), (b), (e), (f), (g)
4. Give a formal proof (using only the rules of inference of predicate logic) of the statement:

\((\exists x, \forall y, P(x, y)) \Rightarrow (\forall y, \exists x, P(x, y))\)

Explain why the converse of this statement is not true. (For more examples of these kinds of formal proofs, see Page 15 of the Lecture Notes.)

Proof.

1. Assume \(\exists x, \forall y, P(x, y)\) -
2. For some \(c\) (constant declaration)
3. \(\forall y, P(c, y)\) \(\exists -; 1\)
4. Let \(d\) be arbitrary -
5. \(P(c, d)\) \(\forall -; 3, 4\)
6. \(\exists x, P(x, d)\) \(\exists +; 2, 5\)
7. -
8. \(\forall y, \exists x, P(x, y)\) \(\forall +; 4, 6, 7\)
9. -
10. \((\exists x, \forall y, P(x, y)) \Rightarrow (\forall y, \exists x, P(x, y))\) \(\Rightarrow +; 1, 8, 9\)

5. (2 points) Give a formal proof (using only predicate logic and equality) of the statement:

\(\exists! x, x = 1\)

6. (3 points) Give a formal proof (using only predicate logic and equality) of the transitive property of equality:

\(\forall x, \forall y, \forall z, (x = y and y = z) \Rightarrow x = z\)

Proof.

1. Let \(a\) be arbitrary -
2. Let \(b\) be arbitrary -
3. Let \(c\) be arbitrary -
4. Assume \(a = b\) and \(b = c\) -
5. \(a = b\) and \(\sim; 4\)
6. \(b = c\) and \(\sim; 4\)
7. \(a = c\) substitution; 5, 6
8. -
9. \((a = b \text{ and } b = c) \Rightarrow a = c \quad \Rightarrow +; 4,7,8\)

10. \(-\)

11. \(\forall z, (a = b \text{ and } b = z) \Rightarrow a = z \quad \forall +; 3,9,10\)

12. \(-\)

13. \(\forall y, \forall z, (a = y \text{ and } y = z) \Rightarrow a = z \quad \forall +; 2,11,12\)

14. \(-\)

15. \(\forall x, \forall y, \forall z, (x = y \text{ and } y = z) \Rightarrow x = z \quad \forall +; 1,13,14\)

Extra practice problems

These are not to be handed in - they are only for your own practice!

1. Section 1.6, exercises 7, 8, and 9.

Bonus Problem

Starting from only the Peano Axioms, prove that addition is associative, that is, \(\forall n, \forall m, n + m = m + n\).

(Hint: first prove that \(\forall m, \sigma(m) = m + 1\), and also prove that \(\forall m, m + 0 = 0 + m = m\).

You can use proof shortcuts, such as if and only if substitution and using the stated axioms as expanded rules of inference. You can also assume that equality is reflexive, symmetric, and transitive, so you can write transitive chains using \(=\).