Variations on a Theme of Schubert Calculus

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Schubert Calculus Quiz: How Schubert-y are you?

0.1. How many lines pass through two given points in the plane $\mathbb{C}^2$?

(a) One
(b) Infinitely many
(c) One, as long as the points are distinct

0.2. How many points are contained in both of two distinct lines in $\mathbb{C}^2$?

(a) None
(b) One
(c) It depends on the choice of lines

0.3. How many planes in $\mathbb{C}^4$ contain a given line and a given point not on the line?

(a) Zero
(b) One
(c) Infinitely many

0.4. How many lines intersect four given lines in general position in $\mathbb{C}^3$?

(a) One
(b) Two
(c) Infinitely many

0.5. How many k-dimensional subspaces of $\mathbb{C}^n$ intersect each of $k \cdot (n-k)$ subspaces of dimension $n-k$ nontrivially?

(a) $\binom{n+k-1}{k}$
(b) $\frac{(k(n-k))! \cdot (k-1)! \cdot (k-2)! \cdot \cdots \cdot 1!}{(n-1)! \cdot (n-2)! \cdot \cdots \cdot 1!}$
(c) It depends on the choice of given subspaces

1 Introduction and background

Schubert's 19th century solution to problem 0.4 on the quiz would have invoked what he called the "Principle of Conservation of Number," as follows. Suppose the four lines l_1, l_2, l_3, l_4 were arranged so that l_1 and l_2 intersect at a point P, l_2 and l_3 intersect at Q, and none of the other pairs of lines intersect and the planes ρ_1 and ρ_2 determined by l_1, l_2 and l_3, l_4 respectively are not parallel. Then ρ_1 and ρ_2 intersect at another line α, which necessarily intersects all four lines. The line β through P and Q also intersects all four lines, and it is not hard to see that these are the only two in this case.

Schubert would have said that since there are two solutions in this configuration and it is a finite number of solutions, it is true for every configuration of lines for which the number is finite by continuity. Unfortunately, due to degenerate cases involving counting with multiplicity, this led to many errors in computations in harder questions of enumerative geometry. Hilbert's 15th problem asked to put Schubert's enumerative methods on a rigorous foundation. This led to the modern-day theory known as Schubert calculus.

The main idea is, taking example 0.4 again, is to let X_i be the space of all lines L intersecting l_i for each $i = 1, \ldots, 4$. Then the intersection $X_1 \cap X_2 \cap X_3 \cap X_4$ is the set of solutions to our problem. Each X_i is an example of a *Schubert variety*, an algebraic and geometric object that is essential to solving these types of intersection problems.

To begin defining these Schubert varieties, we first need to introduce the notions of projective varieties and Grassmannians.

1.1 Projective space

The notion of *projective space* helps clean up many of the ambiguities in the question above. For instance, in the projective plane, parallel lines meet, at a "point at infinity".[1] It also is one of the simplest examples of a Schubert variety.

Figure 1: Parallel lines meeting at a point at infinity.

One way to define projective space over a field k (we'll usually be working with the complex numbers $\mathbb{C}$ here) is as the set of lines through the origin in one higher dimensional space. More rigorously:

Definition 1. *The n-dimensional* ***projective space*** $\mathbb{P}^n_k$ *over a field k is the set of equivalence*

[1]Photo of the train tracks courtesy of edupic.net.

classes

$$\{(x_0, x_1, \ldots, x_n) \in k^{n+1} \setminus \{(0, 0, \ldots, 0)\}\}/\sim$$

where $\sim$ is the equivalence relation given by scalar multiplication:

$$(x_0, x_1, \ldots, x_n) \sim (y_0, y_1, \ldots, y_n)$$

if and only if there exists $a \in k \setminus \{0\}$ such that $ax_i = y_i$ for all i.

We write $(x_0 : x_1 : \cdots : x_n)$ for the point in $\mathbb{P}^k$ containing $(x_0, \ldots, x_n)$.

Unless we specify otherwise, we'll always use $k = \mathbb{C}$ and simply write $\mathbb{P}^n$ for $\mathbb{P}^n_{\mathbb{C}}$ throughout these notes.

Note that *a point in $\mathbb{P}^n_k$ is a line through the origin in k^{n+1}*. In particular, a line through the origin consists of all scalar multiples of a given nonzero vector.

Example 1. In the "projective plane" $\mathbb{P}^2$, the symbols $(2 : 0 : 1)$ and $(4 : 0 : 2)$ both refer to the same point.

It is useful to think of projective space as having its own geometric structure, rather than just as a quotient of a higher dimensional space. In particular, a **geometry** is often defined as a set along with a group of transformations. A **projective transformation** is a map $f : \mathbb{P}^n \to \mathbb{P}^n$ of the form

$$f(x_0 : x_1 : \cdots : x_n) = (y_0 : y_1 : \cdots : y_n)$$

where for each i,

$$y_i = a_{i0}x_0 + a_{i1}x_1 + \cdots a_{in}x_n$$

for some fixed constants $a_{ij} \in \mathbb{C}$ such that the $(n + 1) \times (n + 1)$ matrix (a_{ij}) is invertible.

Notice that projective transformations are well-defined on $\mathbb{P}^n$ because scaling all the x_i variables by a constant c has the effect of scaling the y variables by c as well. This is due to the fact that the defining equations are **homogeneous**: every monomial on both sides of the equation has a fixed degree d (in this case $d = 1$).

1.2 Affine patches and projective varieties

There is another way of thinking of projective space: as ordinary Euclidean space with extra smaller spaces placed out at infinity. For instance, in $\mathbb{P}^1$, any point $(x : y)$ with $y \neq 0$ can be rescaled to the form $(t : 1)$. All such points can be identified with the element $t \in \mathbb{C}$, and then there is only one more point in $\mathbb{P}^1$, namely $(1 : 0)$. We can think of $(1 : 0)$ as a point "at infinity" that closes up the *affine line* $\mathbb{C}^1$ into the "circle" $\mathbb{P}^1$. (Thought of as a real surface, the complex $\mathbb{P}^1$ is actually a sphere.)

Similarly, we can instead parameterize the points $(1 : t)$ by $t \in \mathbb{C}^1$ and have $(0 : 1)$ be the extra point. The subspaces given by $\{(1 : t)\}$ and $\{(t : 1)\}$ are both called **affine patches** of $\mathbb{P}^1$, and form a cover of $\mathbb{P}^1$, from which we can inherit a natural topology on $\mathbb{P}^1$ from the Euclidean topology on each $\mathbb{C}^1$. In fact, the two affine patches form an open cover in this topology, so $\mathbb{P}^1$ is compact.

As another example, the projective plane $\mathbb{P}^2$ can be written as the disjoint union

$$\{(x : y : 1)\} \sqcup \{(x : 1 : 0)\} \sqcup \{1 : 0 : 0\} = \mathbb{C}^2 \sqcup \mathbb{C}^1 \sqcup \mathbb{C}^0,$$

which we can think of as a certain closure of the affine patch $\{(x : y : 1)\}$. The other affine patches are $\{(x : 1 : y)\}$ and $\{(1 : x : y)\}$ in this case.

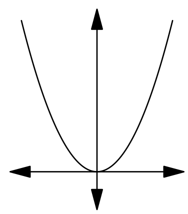
We can naturally generalize this as follows.

Definition 2. *The standard* ***affine patches*** *of* $\mathbb{P}^n$ *are the sets*

$$\{(t_0 : t_1 : \cdots : t_{i-1} : 1 : t_{i+1} : \cdots : t_n)\} \cong \mathbb{C}^n$$

for $i = 0, \ldots, n$.

An **affine variety** is usually defined as the vanishing locus of a set of polynomials in k^n for some field k. For instance, the graph of $y = x^2$ is an affine variety, since it is the set of all points (x, y) for which $f(x, y) = y - x^2$ is zero.



And in three-dimensional space, we might consider the plane defined by the zero locus of $f(x, y, z) = x + y + z$, or the line $x = y = z$ defined by the common zero locus of $f(x, y, z) = x - y$ and $g(x, y, z) = x - z$.

Definition 3. *A* ***projective variety*** *is the common zero locus in* $\mathbb{P}^n$ *of a finite set of homogeneous polynomials* $f_1(x_0, \ldots, x_n), \ldots, f_r(x_0, \ldots, x_n)$ *in* $\mathbb{P}^n$. *We call this variety* $V(f_1, \ldots, f_r)$. *In other words,*

$$V(f_1, \ldots, f_r) = \{(a_0 : \cdots : a_n) \mid f_i(a_0 : \cdots : a_n) = 0 \text{ for all } i\}.$$

Remark 1. Recall that a polynomial is **homogeneous** if all of its terms have the same total degree. Note that we need the homogeneous condition in order for projective varieties to be well-defined. For instance, if $f(x, y) = y - x^2$ then $f(2 : 4) = 0$ and $f(4 : 8) \neq 0$, but $(2 : 4) = (4 : 8)$ in $\mathbb{P}^1$. So the value of a nonhomogeneous polynomial on a point in projective space is not, in general, well-defined.

The intersection of a projective variety with the i-th affine patch is the *affine* variety formed by setting $x_i = 1$ in all of the defining equations. For instance, the projective variety in $\mathbb{P}^2$ defined by $f(x : y : z) = yz - x^2$ restricts to the affine variety defined by $f(x, y) = y - x^2$ in the affine patch $z = 1$.

We can also reverse this process. The **homogenization** of a polynomial $f(x_0, \ldots, x_{n-1})$ in n variables using another variable x_n is the unique homogeneous polynomial $g(x_0 : \cdots : x_{n-1} : x_n)$ with $\deg(g) = \deg(f)$ for which

$$g(x_0 : \cdots : x_{n-1} : 1) = f(x_0, \ldots, x_{n-1}).$$

For instance, the homogenization of $y - x^2$ is $yz - x^2$. If we homogenize the equations of an affine variety, we get a projective variety which we call its **projective closure**.

Example 2. The projective closure of the parabola defined by $y - x^2 - 1 = 0$ is the projective variety in $\mathbb{P}^3$ defined by the equation $yz - x^2 - z^2 = 0$. If we intersect this with the $y = 1$ affine patch, we obtain the affine variety $z - x^2 - z^2 = 0$ in the x, z variables. This is the circle $x^2 + (z - \frac{1}{2})^2 = \frac{1}{4}$, and so parabolas and circles are essentially the same object in projective space, cut different ways into affine patches.

As explained in more detail in the problems below, there is only one type of (nondegenerate) conic in projective space!

Remark 2. The above example implies that if we draw a parabola on a large, flat plane and stand at its apex, looking out to the horizon we will see the two branches of the parabola meeting at a point on the horizon, closing up the curve into an ellipse.[2]

1.3 Points, lines, and m-planes in projective space

Just as the points of $\mathbb{P}^n$ are the images of lines in $\mathbb{C}^{n+1}$, a **line** in projective space can be defined as the image of a **plane** in k^{n+1}, and so on. We can define these in terms of homogeneous coordinates as follows.

Definition 4. *An $(n-1)$-**plane** or **hyperplane** in $\mathbb{P}^n$ is the set of solutions $(x_0 : \cdots : x_n)$ to a homogeneous linear equation*

$$a_0x_0 + a_1x_1 + \cdots + a_nx_n = 0.$$

*A k-**plane** is an intersection of $n-k$ hyperplanes, say $a_{i0}x_0 + a_{i1}x_1 + \cdots + a_{in}x_n = 0$ for $i = 1, \ldots, n-k$, such that the matrix of coefficients (a_{ij}) is full rank.*

Example 3. In the projective plane $\mathbb{P}^n$, the line l_1 given by $2x + 3y + z = 0$ restricts to the line $2x + 3y + 1 = 0$ in the affine patch $z = 1$. Notice that the line l_2 given by $2x + 3y + 2z = 0$ restricts to $2x + 3y + 2 = 0$ in this affine patch, and is parallel to the restriction of l_1 in this patch. However, the projective closures of these affine lines intersect at the point $(3 : -2 : 0)$, on the $z = 0$ line at infinity.

In fact, any two distinct lines meet in a point in the projective plane. In general, intersection problems are much easier in projective space. See Problem 1.4 below to apply this to our problems in Schubert Calculus.

1.4 Problems

1.1. **Transformations of $\mathbb{P}^1$:** Show that a projective transformation on $\mathbb{P}^1$ is uniquely determined by where it sends $0 = (0 : 1)$, $1 = (1 : 1)$, and $\infty = (1 : 0)$.

1.2. **Choice of $n+2$ points stabilizes $\mathbb{P}^n$:** Construct a set S of $n+2$ distinct points in $\mathbb{P}^n$ for which any projective transformation is uniquely determined by where it sends each point of S. What are necessary and sufficient conditions for a set of $n+2$ distinct points in $\mathbb{P}^n$ to have this property?

1.3. **All conics in $\mathbb{P}^2$ are the same:** Show that, for any quadratic homogeneous polynomial $f(x, y, z)$ there is a projective transformation that sends it to one of x^2, x^2+y^2, or $x^2+y^2+z^2$. Conclude that any two "nondegenerate" conics are the same up to a projective transformation.

   (Hint: Any quadratic form can be written as $\mathbf{x}A\mathbf{x}^T$ where $\mathbf{x} = (x, y, z)$ is the row vector of variables and $\mathbf{x}^T$ is its transpose, and A is a symmetric matrix, with $A = A^T$. It can be shown that a symmetric matrix A can be diagonalized, i.e., written as BDB^T for some diagonal matrix D. Use the matrix B as a projective transformation to write the quadratic form as a sum of squares.)

1.4. **Schubert Calculus Quiz V2:** In the Schubert Calculus Quiz, problem 0.2 can be projectivized as follows: if we ask instead how many points are contained in two distinct lines in $\mathbb{P}^2$, then the answer is (b) rather than (c), a much nicer answer! Write out projective versions of each of the five problems. What do they translate to in terms of intersections of subspaces of one-higher-dimensional affine space?

[2]Unfortunately, we could not find any photographs of parabolic train tracks.

2 Theme: The Grassmannian

Not only does taking the projective closure of our problems in $\mathbb{P}^n$ make things easier, it is also useful to think of the intersection problems as involving subspaces of $\mathbb{C}^{n+1}$ rather than k-planes in $\mathbb{P}^n$.

The following definition is exactly analogous to our first definition of projective space.

Definition 5. *The* ***Grassmannian*** $\mathrm{Gr}(n,k)$ *is the set of all dimension-k subspaces of* $\mathbb{C}^n$*.*

As in projective spaces, we call the elements of $\mathrm{Gr}(n,k)$ the "points" of $\mathrm{Gr}(n,k)$, even though they are defined as entire subspaces of $\mathbb{C}^n$. We will see soon that $\mathrm{Gr}(n,k)$ has the structure of a projective variety, making this notation useful.

Every point of the Grassmannian can be described as the span of some k independent row vectors of length n, which we can arrange in a $k \times n$ matrix. For instance, the following represents a point in $\mathrm{Gr}(7,3)$.

$$\begin{bmatrix} 0 & -1 & -3 & -1 & 6 & -4 & 5 \\ 0 & 1 & 3 & 2 & -7 & 6 & -5 \\ 0 & 0 & 0 & 2 & -2 & 4 & -2 \end{bmatrix}$$

Notice that we can perform elementary row operations on the matrix without changing the point of the Grassmannian it represents. Let's use the convention that the pivots will be in order from left to right and bottom to top.

Exercise 1. Show that the matrix above has reduced row echelon form:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 1 & * & 0 & * & * & 0 \end{bmatrix},$$

where the $*$ entries can be any complex numbers.

We can summarize our findings as follows.

Fact 1. *Each point of* $\mathrm{Gr}(n,k)$ *is the row span of a unique full-rank $k \times n$ matrix in reduced row echelon form.*

The subset of the Grassmannian whose points have this particular reduced row echelon form constitutes a **Schubert cell.** Notice that $\mathrm{Gr}(n,k)$ is a disjoint union of Schubert cells.

2.1 Projective variety structure

The Grassmannian can be viewed as a projective variety by embedding $\mathrm{Gr}(n,k)$ in $\mathbb{P}^{\binom{n}{k}-1}$ via the *Plücker embedding.* To do so, choose an ordering on the k-element subsets S of $\{1,2,\ldots,n\}$ and use this ordering to label the homogeneous coordinates x_S of $\mathbb{P}^{\binom{m}{r}-1}$. Now, given a point in the Grassmannian represented by a matrix M, let x_S be the determinant of the $r \times r$ submatrix determined by the columns in the subset S. This determines a point in projective space since row operations can only change the determinants up to a constant factor, and the coordinates cannot all be zero since the matrix has rank r.

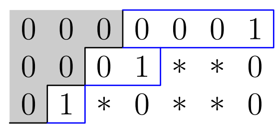
One can show that the image is a projective variety in $\mathbb{P}^{\binom{m}{r}-1}$, cut out by homogeneous quadratic relations known as the *Plücker relations.* (See [1], pg. 408.)

2.2 Schubert cells and Schubert varieties

To enumerate the Schubert cells in the Grassmannian, we assign to the matrices of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 1 & * & 0 & * & * & 0 \end{bmatrix}$$

a **partition**, that is, a nonincreasing sequence of nonnegative integers $\lambda = (\lambda_1, \ldots, \lambda_k)$, as follows. Cut out the $k \times k$ staircase from the upper left corner of the matrix, and let λ_i be the distance from the edge of the staircase to the 1 in row i. In the example shown, we get the partition $\lambda = (4, 2, 1)$. Notice that we always have $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$.

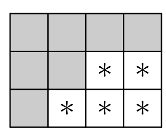


Remark 3. We say the **size** of a partition λ, denoted $|\lambda|$, is $\sum_i \lambda_i$, and its **length**, denoted $l(\lambda)$, is the number of nonzero parts. The entries λ_i are called its **parts**, and its **longest part** is λ_1.

With this notation, Schubert cells in $\mathrm{Gr}(n, k)$ are in bijection with the partitions λ for which $l(\lambda) \leq k$ and $\lambda_1 \leq n - k$.

Definition 6. *The **Young diagram** of a partition λ is the left-aligned partial grid of boxes in which the i-th row from the top has λ_i boxes.*

For example, the Young diagram of the partition $(4, 2, 1)$ that came up in the previous example is shown as the shaded boxes in the diagram below. By identifying the partition with its Young diagram, we can alternatively define λ as the complement in a $k \times (n - k)$ rectangle of the diagram μ defined by the right-aligned shift of the $*$ entries in the matrix:



Since the $k \times (n - k)$ rectangle is the bounding shape of our allowable partitions, we will call it the **Important Box**. (Warning: this terminology is not standard.)

Definition 7. *For a partition λ contained in the Important Box, the **Schubert cell** Ω_λ° is the set of points of $\mathrm{Gr}(n, k)$ whose row echelon matrix has corresponding partition λ. Explicitly,*

$$\Omega_\lambda^\circ = \{V \in \mathrm{Gr}(n, k) \mid \dim V \cap \langle e_1, \ldots, e_r \rangle = i \text{ for } n - k + i - \lambda_i \leq r \leq n - k + i - \lambda_{i+1}\}.$$

Here e_{n-i} is the i-th standard unit vector $(0, 0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 in the i-th position.

Since each $*$ can be any complex number, we have $\Omega_\lambda^\circ \cong \mathbb{C}^{k(n-k)-|\lambda|}$, and so

$$\dim \Omega_\lambda^\circ = k(n - k) - |\lambda|.$$

In particular the dimension of the Grassmannian is $k(n - k)$.

We are now in a position to define **Schubert varieties** as closed subvarieties of the Grassmannian (the intersection of $\mathrm{Gr}(n, k)$ with another projective variety in the ambient space).

Definition 8. *The **standard Schubert variety** corresponding to a partition λ, denoted Ω_λ, is the closure $\overline{\Omega_\lambda{}^\circ}$ of the corresponding Schubert cell. Explicitly,*

$$\Omega_\lambda = \{V \in \mathrm{Gr}(n,k) \mid \dim V \cap \langle e_1, \ldots, e_{n-k+i-\lambda_i} \rangle \geq i\}.$$

See the Problems section below to understand why this gives the closure. Note that we have $\dim \Omega_\lambda = \dim \Omega_\lambda^\circ = k(n-k) - |\lambda|$ as well.

Example 4. Consider the Schubert variety $\Omega_{\square\square}$ in $\mathbb{P}^5 = \mathrm{Gr}(6,1)$. The Important Box is a 1×5 row of squares. There is one condition defining the points V in this variety, namely $\dim V \cap \langle e_1, e_2, e_3, e_4 \rangle \geq 1$, where V is a one-dimensional subspace of $\mathbb{C}^6$. This means that V is contained in the span of $e_1, \ldots, e_4$, and so, expressed in homogeneous coordinates, its first two entries (in positions e_5 and e_6) are 0.

Thus each point of $\Omega_{\square\square}$ can be written in one of the following forms:

$$\begin{array}{l}(0:0:1:*:*:*)\\(0:0:0:1:*:*)\\(0:0:0:0:1:*)\\(0:0:0:0:0:1)\end{array}$$

It follows that $\Omega_{\square\square}$ can be written as a disjoint union of Schubert cells as follows:

$$\Omega_{\square\square} = \Omega^\circ_{\square\square} \sqcup \Omega^\circ_{\square\square\square} \sqcup \Omega^\circ_{\square\square\square\square} \sqcup \Omega^\circ_{\square\square\square\square\square}.$$

In fact, every Schubert variety is a disjoing union of Schubert cells (see the Problems below).

In general, we can use a different basis than the standard basis $e_1, \ldots, e_n$ for $\mathbb{C}^n$ in order to construct a Schubert variety. Given a **complete flag**, i.e. a chain of subspaces

$$0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n$$

where each F_i has dimension i, we define

$$\Omega_\lambda(F_\bullet) = \{V \in \mathrm{Gr}(n,k) \mid \dim V \cap F_{n-k+i-\lambda_i} \geq i\}$$

and similarly for Ω_λ°.

Remark 4. The numbers $n-k+i-\lambda_i$ are the positions of the 1's in the matrix counted from the right.

Example 5. The Schubert variety $\Omega_\square(F_\bullet) \subset \mathrm{Gr}(4,2)$ consists of the 2-dimensional subspaces V of $\mathbb{C}^4$ for which $\dim V \cap F_2 \geq 1$. We can express this in terms of $\mathbb{P}^3$ as the space of all lines in $\mathbb{P}^3$ that intersect a given line (the image of F_2 in the quotient) in at least a point. This is precisely the variety we need to analyze Problem 0.4 on the Schubert Calculus Quiz.

2.3 A note on flags

Why are chains of subspaces called *flags*? Roughly speaking, a flag on a flagpole consists of:

- A point (the bulbous part at the top of the pole),
- A line passing through that point (the pole),
- A plane passing through that line (the plane containing the flag), and

- Space to put it in.

Mathematically, this is the data of a *complete flag* in three dimensions. However, higher-dimensional beings would require more complicated flags. So in general, it is natural to define a complete flag in n-dimensional space $\mathbb{C}^n$ to be a chain of vector spaces F_i of each dimension from 0 to n, each containing the previous, with $\dim F_i = i$ for all i. A **partial flag** is a chain of subspaces in which only some of the possible dimensions are included.

2.4 Problems

2.1. **Projective space is a Grassmannian:** Show that every projective space $\mathbb{P}^m$ is a Grassmannian. What are n and k?

2.2. **Schubert cells in $\mathbb{P}^m$:** What are the Schubert cells in $\mathbb{P}^m$? Express your answer in homogeneous coordinates.

2.3. **Schubert varieties in $\mathbb{P}^m$:** What are the Schubert varieties in $\mathbb{P}^m$, thought of as a Grassmannian? Why are they the closures of the Schubert cells in the topology on $\mathbb{P}^m$?

2.4. **Schubert varieties vs. Schubert cells:** Show that every Schubert variety is a disjoint union of Schubert cells. Which Schubert cells are contained in Ω_λ?

2.5. **Extreme cases:** Describe $\Omega_\emptyset$ and Ω_B where B is the entire Important Box. What are their dimensions?

2.6. **Intersecting Schubert Varieties:** Show that, by choosing four different flags $F_\bullet^{(1)}$, $F_\bullet^{(2)}$, $F_\bullet^{(3)}$, $F_\bullet^{(4)}$, Problem 0.4 on the Schubert Calculus Quiz becomes equivalent to finding the intersection of the Schubert varieties

$$\Omega_\square(F_\bullet^{(1)}) \cap \Omega_\square(F_\bullet^{(2)}) \cap \Omega_\square(F_\bullet^{(3)}) \cap \Omega_\square(F_\bullet^{(4)}).$$

2.7. **Schubert Calculus Quiz, V3:** Translate each one of the problems on the projectivized version of the Schubert Calculus Quiz into a problem about intersections of Schubert varieties, as we did for Problem 0.4 in problem 2.6 above.

2.8. **More complicated flag conditions:** In $\mathbb{P}^4$, 2-planes A and B intersect in a point X, and P and Q are distinct points different from X. Express the set of all 2-planes C that contain both P and Q and intersect A and B each in a line as an intersection of Schubert varieties in $\mathrm{Gr}(5,3)$, in each of the following cases:

   (a) When P is contained in A and Q is contained in B;

   (b) When neither P nor Q lie on A or B.

3 Variation 1: Intersections of Schubert varieties in the Grassmannian

In the previous section, we saw how to express certain linear intersection problems as intersections of Schubert varieties in a Grassmannian. We now will build up the machinery needed to obtain a combinatorial rule for computing these intersections known as the **Littlewood-Richardson rule**.

Unfortunately, both the geometric and combinatorial aspects of the Littlewood-Richardson rule are too complicated to fully prove by the end of this mini-course. In fact, we do not even yet have the tools to state the full version of the rule, so we will start by stating the "zero-dimensional" rule, the case when the intersection of the Schubert varieties is zero-dimensional.

Theorem 1 (Zero-dimensional Littlewood-Richardson rule.)**.** *Let n, k be positive integers with $k \leq n$. Given a list of generic flags $F_\bullet^{(i)}$ in $\mathbb{C}^n$ for $i = 1, \ldots, r$, let $\lambda^{(1)}, \ldots, \lambda^{(r)}$ be partitions with*

$$\sum |\lambda^i| = k(n-k).$$

Then the intersection

$$\bigcap \Omega_{\lambda^i}(F_\bullet^{(i)})$$

*is zero-dimensional and consists of exactly $c^B_{\lambda^{(1)},\ldots,\lambda^{(r)}}$ points of $\mathrm{Gr}(n,k)$, where B is the Important Box and $c^B_{\lambda^{(1)},\ldots,\lambda^{(r)}}$ is a certain **Littlewood-Richardson coefficient**.*

When we say a "generic" choice of flags, we mean that in the space of all tuples of flags, the theorem is true on an open dense subset. This will be made more precise in the next section when we discuss flag varieties.

To gain intuition for zero-dimensional intersections, it helps to simplify even further: to the case of two flags that intersect in a "transverse" way.

3.1 Opposite and transverse flags, genericity

Two subspaces of $\mathbb{C}^n$ are said to be *transverse* if their intersection has the "expected dimension". For instance, two 2-dimensional subspaces of $\mathbb{C}^3$ are expected to have a 1-dimensional intersection; only rarely is their intersection 2-dimensional (when the two planes coincide). More rigorously:

Definition 9. *Two subspaces V and W of $\mathbb{C}^n$ are **transverse** if*

$$\dim(V \cap W) = \max(0, \dim V + \dim W - n).$$

Equivalently, if $\mathrm{codim}(V)$ is defined to be $n - \dim(V)$,

$$\mathrm{codim}(V \cap W) = \min(n, \mathrm{codim}(V) + \mathrm{codim}(W)).$$

Exercise 2. Verify that the two definitions above are equivalent.

We say two flags $F_\bullet^{(1)}$ and $F_\bullet^{(2)}$ are **transverse** if every pair of subspaces $F_i^{(1)}$ and $F_j^{(2)}$ are transverse. In fact, a weaker condition suffices:

Lemma 1. *Two complete flags $F_\bullet, E_\bullet \subset \mathbb{C}^n$ are transverse if and only if $F_{n-i} \cap E_i = 0$ for all i.*

Proof Sketch. The forward direction is clear. For the reverse implication, we can take the quotient of both flags by the one-dimensional subspace E_1 and induct on n. □

An important example of transverse flags are the **standard flag** in which $F_i = \langle e_1, \dots, e_i \rangle$ and the **opposite flag** in which $E_i = \langle e_n, \dots, e_{n-i+1} \rangle$. It is easy to check that these flags $F_\bullet$ and $E_\bullet$ are transverse. Furthermore, it turns out that every pair of transverse flags can be mapped to this pair, as follows. Consider the action of $\mathrm{GL}_n(\mathbb{C})$ on $\mathbb{C}^n$ by standard matrix multiplication, and note that this gives rise to an action on flags and subspaces, and subsequently on Schubert varieties as well.

Lemma 2. *For any pair of transverse flags $F'_\bullet$ and $E'_\bullet$, there is an element $g \in \mathrm{GL}_n$ such that $gF'_\bullet = F_\bullet$ and $gE'_\bullet = E_\bullet$, where $F_\bullet$ and $E_\bullet$ are the standard and opposite flags.*

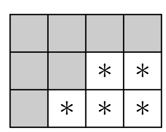
The proof of this lemma is left as an exercise to the reader (see the Problems section below). The important corollary is that to understand the intersection of the Schubert varieties $\Omega_\lambda(F'_\bullet)$ and $\Omega_\mu(E'_\bullet)$, it suffices to compute the intersection $\Omega_\lambda(F_\bullet) \cap \Omega_\lambda(E_\bullet)$ and multiply the results by the appropriate matrix g.

So, when we consider the intersection of two Schubert varieties with respect to transverse flags, it suffices to consider the standard and opposite flags $F_\bullet$ and $E_\bullet$. We use this principle in the **duality theorem** below, which tells us exactly when the intersection of $\Omega_\lambda(F_\bullet)$ and $\Omega_\mu(E_\bullet)$ is nonempty.

3.2 Duality theorem

Definition 10. *Two partitions $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_k)$ (where some of the parts may be 0) are **complimentary** in the $k \times (n-k)$ Important Box if and only if $\lambda_i + \mu_{k+1-i} = n - k$ for all i. In this case we write $\mu^c = \lambda$.*

In other words, if we rotate the Young diagram of μ and place it in the lower right corner of the Important Box, its complement is λ. Below, we see that $\mu = (3, 2)$ is the complement of $\lambda = (4, 2, 1)$ in $\mathrm{Gr}(7, 3)$.



Theorem 2 (Duality Theorem)**.** *Let $F_\bullet$ and $E_\bullet$ be transverse flags in $\mathbb{C}^n$, and let λ and μ be partitions with $|\lambda| + |\mu| = k(n-k)$. In $\mathrm{Gr}(n, k)$, the intersection $\Omega_\lambda(F_\bullet) \cap \Omega_\mu(E_\bullet)$ has 1 element if μ and λ are complimentary partitions, and is empty otherwise.*

Furthermore, if μ and λ are any partitions with $\mu_{k+1-i} + \lambda_i > n - k$ for some i then $\Omega_\lambda(F_\bullet) \cap \Omega_\mu(E_\bullet) = \emptyset$.

Intuitively, we can use a reversed row reduction to express the Schubert varieties with respect to the opposite flag, and then the Schubert cells for the complementary partitions will have their 1's in the same positions, as in the example below. Their unique intersection will be precisely this matrix of 1's with 0's elsewhere.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 1 & * & 0 & * & * & 0 \end{bmatrix} \qquad \begin{bmatrix} * & 0 & * & 0 & * & * & 1 \\ * & 0 & * & 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We now give a more rigorous proof below.

Proof. We prove the second claim first: if for some i we have $\mu_{k+1-i} + \lambda_i > n - k$ then $\Omega_\lambda(F_\bullet) \cap \Omega_\mu(E_\bullet)$ is empty. Assume for contradiction that there is an element V in the intersection. We know $\dim(V) = k$, and also

$$\dim(V \cap \langle e_1, e_2, \ldots, e_{n-k+i-\lambda_i} \rangle) \geq i, \tag{1}$$

$$\dim(V \cap \langle e_n, e_{n-1}, \ldots, e_{n+1-(n-k+(k+1-i)-\mu_{k+1-i})} \rangle) \geq k + 1 - i.$$

Simplifying the last subscript above, and reversing the order of the generators, we get

$$\dim(V \cap \langle e_{i+\mu_{k+1-i}}, \ldots, e_{n-1}, e_n \rangle) \geq k + 1 - i. \tag{2}$$

Notice that $i + \mu_{k+1-i} > n - k + i - \lambda_i$ by the condition $\mu_{k+1-i} + \lambda_i > n - k$, and so the two subspaces we are intersecting V with in equations (1) and (2) are disjoint. It follows that V has total dimension at least $k + 1 - i + i = k + 1$, a contradiction. Thus $\Omega_\lambda(F_\bullet) \cap \Omega_\mu(E_\bullet)$ is empty in this case.

This implies that if $|\lambda| + |\mu| = k(n - k)$ and they are not complimentary then the intersection is empty as well, since the inequality $\mu_{k+1-i} + \lambda_i > n - k$ must hold for some i.

Finally, suppose λ and μ are complimentary. Then equations (1) and (2) still hold, but now $n - k + i - \lambda_i = i + \mu_{n+1-i}$ for all i. Thus $\dim(V \cap \langle e_{i+\mu_{n+1-i}} \rangle) = 1$ for all $i = 1, \ldots, k$, and since V is k-dimensional it must equal the span of these basis elements, namely

$$V = \langle e_{1+\mu_n}, e_{2+\mu_{n-1}}, \ldots e_{k+\mu_{n+1-k}} \rangle.$$

This is the unique solution. □

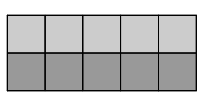
Example 6. We now can give the most high-powered proof imaginable that there is a unique line passing through any two distinct points in $\mathbb{P}^n$. Working in $\mathrm{Gr}(n + 1, 2)$, the two distinct points become two distinct one-dimensional subspaces F_1 and E_1 of $\mathbb{C}^{n+1}$, and the Schubert condition that demands the 2-dimensional subspace V contains them is

$$\dim(V \cap F_1) \geq 1, \qquad \dim(V \cap E_1) \geq 1.$$

These are the Schubert conditions for a single-part partition $\lambda = (\lambda_1)$ where $(n+1)-2+1-\lambda_1 = 1$. Thus $\lambda_1 = n - 1 = (n + 1) - 2$, and we are intersecting the Schubert varieties

$$\Omega_{(n-1)}(F_\bullet) \cap \Omega_{(n-1)}(E_\bullet)$$

where $F_\bullet$ and $E_\bullet$ are any two transverse flags extending F_1 and E_1. Notice that $(n-1)$ and $(n-1)$ are complimentary partitions in the $2 \times (n - 1)$ Important Box, and so by the Duality Theorem there is a unique point of $\mathrm{Gr}(n + 1, 2)$ in the intersection. The conclusion follows.



3.3 Cell complex structure

In order to prove the more general zero-dimensional Littlewood-Richardson rule and compute the Littlewood-Richardson coefficients, we need to develop more heavy machinery. In particular, we need to understand the Grassmannian as a geometric object and compute its **cohomology**, an associated ring in which multiplication of certain generators will correspond to intersection of Schubert varieties.

The term *Schubert cell* comes from the notion of a **cell complex** (also known as a CW complex) in algebraic topology. (See [4] for more details on everything in this section.) An n**-cell** can be defined as a copy of the open ball $|v| < 1$ in $\mathbb{R}^n$, and its associated n**-disk** is its closure $|v| \le 1$ in $\mathbb{R}^n$.

To construct a cell complex, one starts with a set of points called the 0**-skeleton** X^0, and then attaches 1-disks D via continuous boundary maps $\partial D \to X^0$. The result is a 1**-skeleton** X^1, which can then be extended to a 2-skeleton by attaching 2-disks E via maps $\partial E \to X^1$. In general X^n is the union of X^{n-1} with the n-disks quotiented by the attachment maps, and the open n-cells associated to each of the disks cover the cell complex X. The topology is given by $A \subset X$ is open if and only if $A \cap X^n$ is open in X^n for all n.

Example 7. The real projective plane $\mathbb{P}^2_{\mathbb{R}}$ has a cell complex structure in which $X^0 = \{(0:0:1)\}$ is a single point, $X^1 = X^0 \sqcup \{(0:1:*)\}$ is topologically a circle formed by attaching a 1-cell to the point at both ends, and then X^2 is formed by attaching a 2-cell $\mathbb{R}^2$ to the circle such that the boundary wraps around the 1-cell twice. This is because the points of the form $(1 : xt : yt)$ as $t \to \infty$ and as $t \to -\infty$ both approach the same point in X^1, so the boundary map must be a 2-to-1 mapping.

Example 8. The *complex* projective plane $\mathbb{P}^2_{\mathbb{C}}$ has a simpler cell complex structure, consisting of starting with a single point $X^0 = \{(0:0:1)\}$, and then attaching a 2-*cell* (a copy of $\mathbb{C} = \mathbb{R}^2$) like a balloon to form X^2. A copy of $\mathbb{C}^2 = \mathbb{R}^4$ is then attached to form X^4.

The Schubert varieties give a cell complex structure on the Grassmannian. Define the 0-skeleton X_0 to be the 0-dimensional Schubert variety $\Omega_{((n-k)^k)}$. Define X_2 to be X_0 along with the 2-cell (since we are working over $\mathbb{C}$ and not $\mathbb{R}$) formed by removing a corner square from the rectangular partition (n^r), and the attaching map given by the closure in $\mathrm{Gr}(n,k)$. Continue in this manner to define the entire cell structure, $X_0 \subset X_2 \subset \cdots \subset X_{2nr}$.

Example 9. We have

$$\mathrm{Gr}(4,2) = \Omega^\circ_{\emptyset} \sqcup \Omega^\circ_{\ydiagram{1}} \sqcup \Omega^\circ_{\ydiagram{2}} \sqcup \Omega^\circ_{\ydiagram{1,1}} \sqcup \Omega^\circ_{\ydiagram{2,1}} \sqcup \Omega^\circ_{\ydiagram{2,2}},$$

forming a cell complex structure in which $X^0 = \Omega^\circ_{\emptyset}$, X^2 is formed by attaching $\Omega^\circ_{\ydiagram{1}}$, X^4 is formed by attaching $\Omega^\circ_{\ydiagram{2}} \sqcup \Omega^\circ_{\ydiagram{1,1}}$, X^6 is formed by attaching $\Omega^\circ_{\ydiagram{2,1}}$, and X^8 is formed by attaching $\Omega^\circ_{\ydiagram{2,2}}$.

3.4 Cellular homology and cohomology

For a CW complex $X = X^0 \subset \cdots \subset X^n$, define

$$C_k = H_k(X^k, X^{k-1}) = \mathbb{Z}^{\#k\text{-cells}},$$

the free abelian group generated by the k-cells $B^{(k)}_\alpha$.

Define the **cellular boundary map** $d_{k+1} : C_{k+1} \to C_k$ by

$$d_{k+1}(B^{(k+1)}_\alpha) = \sum_\beta \deg_{\alpha\beta} \cdot B^{(k)}_\beta,$$

where $\deg_{\alpha\beta}$ is the **degree** of the composite map

$$\partial \overline{B^{(k+1)}_\alpha} \to X^k \to \overline{B^{(k)}_\beta}.$$

The first map above is the cellular attaching map from the boundary of the closure of the ball $B_\alpha^{(k+1)}$ to the k-skeleton, and the second map is the quotient map formed by collapsing $X^k \setminus B_\beta^{(k)}$ to a point. The composite is a map from a k-sphere to another k-sphere, which has a **degree**, whose precise definition we omit here and refer the reader to [4] for details. As one example, the 2-to-1 attaching map described in Example 7 for $\mathbb{P}_{\mathbb{R}}^2$ has degree 2.

Example 10. Recall that $\mathbb{P}_{\mathbb{C}}^2$ consists of a point, a 2-cell attached to it, and then a 4-cell attached to that. So, its cellular chain complex is:

$$\cdots \to 0 \to 0 \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0$$

and the homology groups are $H_0 = H_2 = H_4 = \mathbb{Z}$, $H_1 = H_3 = 0$.

On the other hand, in $\mathbb{P}_{\mathbb{R}}^2$, the chain complex looks like:

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0$$

where the first map $\mathbb{Z} \to \mathbb{Z}$ is multiplication by 2 and the second is the 0 map, due to the degrees of the attaching maps. It follows that $H_2 = 0$, $H_1 = \mathbb{Z}/2\,\mathbb{Z}$, and $H_0 = \mathbb{Z}$.

It is known that the cellular boundary maps make the groups C_k into a **chain complex**: a sequence of maps

$$0 \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \to \cdots \to C_1 \xrightarrow{d_1} C_0 \to 0$$

for which $d_i \circ d_{i+1} = 0$ for all i. This latter condition implies that the image of the map d_{i+1} is contained in the kernel of d_i for all i, and so we can consider the quotient groups

$$H_i(X) = \ker(d_i)/\operatorname{im}(d_{i+1})$$

for all i. These groups are called the **cellular homology groups** of the space X.

We can now define the **cellular cohomology** by dualizing the chain complex above. In particular, define $C^k = \operatorname{Hom}(C_k, \mathbb{Z})$ for each k, and define the **coboundary maps** $d_k^* : C^{k-1} \to C^k$ by

$$d_k^* f(c) = f(d_k(c))$$

for any $f \in C^k$ and $c \in C_k$. Then the coboundary maps form a **cochain complex**, and we can define the cohomology groups

$$H^i(X) = \ker(d_{i+1}^*)/\operatorname{im}(d_i^*)$$

for all i.

Example 11. The cellular cochain complex for $\mathbb{P}_{\mathbb{C}}^2$ is

$$0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to 0 \to 0 \to \cdots$$

and so the cohomology groups are $H^0 = H^2 = H^4 = \mathbb{Z}$, $H^1 = H^3 = 0$.

Finally, the direct sum of the cohomology groups

$$H^*(X) = \oplus H^i(X)$$

has a ring structure given by the **cup product**, which is the dual of the "cap product" on homology and roughly corresponds to taking intersection of cohomology classes. (See [4] for full definitions of these operations.)

In particular, there is an equivalent definition of cohomology on smooth varieties (like the Grassmannian) in which cohomology classes in $H^*(X)$ are equivalence classes algebraic subvarieties under **rational equivalence**. In other words, deformations under rational families are still equivalent: in $\mathbb{P}^2$, for instance, the family of algebraic subvarieties of the form $xy - tz^2 = 0$ as $t \in \mathbb{C}$ varies are all in one equivalence class, even as $t \to 0$ and the hyperbola degenerates into two lines.

The main fact we will be using under this interpretation is the following, which we state without proof. See [2] for more details.

Theorem 3. *The cohomology ring $H^*(\mathrm{Gr}(n,k))$ has a $\mathbb{Z}$-basis given by the classes*

$$\sigma_\lambda := [\Omega_\lambda(F_\bullet)] \in H^{2|\lambda|}(\mathrm{Gr}(n,k))$$

for λ a partition fitting inside the Important Box. The cohomology with cup product is a graded ring, so $\sigma_\lambda \cdot \sigma_\mu \in H^{2|\lambda|+2|\mu|}(\mathrm{Gr}(n,k))$, and we have

$$\sigma_\lambda \cdot \sigma_\mu = [\Omega_\lambda(F_\bullet) \cap \Omega_\mu(E_\bullet)]$$

where $F_\bullet$ and $E_\bullet$ are the standard and opposite flags.

Note that σ_λ is independent of the choice of flag $F_\bullet$, since any two Schubert varieties of the same partition shape are rationally equivalent via a change of basis.

Therefore, we can now restate our intersection problems in terms of multiplying Schubert varieties. In particular, if $\lambda^{(1)}, \ldots, \lambda^{(r)}$ are partitions with $\sum_i |\lambda^{(i)}| = k(n-k)$, then

$$\sigma_{\lambda^{(1)}} \cdots \sigma_{\lambda^{(r)}} \in H^{k(n-k)}(\mathrm{Gr}(n,k))$$

and there is only one generator of the top cohomology group, namely σ_B where B is the important box. This is the cohomology class of the single point $\Omega_B(F_\bullet)$ for some flag $F_\bullet$. Thus the intersection of the Schubert varieties $\Omega_{\lambda^{(1)}}(F_\bullet^{(1)}) \cap \cdots \cap \Omega_{\lambda^{(r)}}(F_\bullet^{(r)})$ is rationally equivalent to a finite union of points, the number of which is the coefficient $c = c^B_{\lambda^{(1)},\ldots,\lambda^{(r)}}$ in the expansion

$$\sigma_{\lambda^{(1)}} \cdots \sigma_{\lambda^{(r)}} = c^B_{\lambda^{(1)},\ldots,\lambda^{(r)}} \sigma_B.$$

And for a sufficiently general choice of flags, the c points in the intersection are distinct with no multiplicity.

So, in general, we wish to understand the coefficients that we get upon multiplying Schubert classes and expressing the product back in the basis $\{\sigma_\lambda\}$ of Schubert classes.

3.5 Connection with symmetric functions

It turns out that we can describe the cohomology ring $H^*(\mathrm{Gr}(n,k))$ explicity as a quotient of the ring of **symmetric functions**.

Definition 11. *The ring of symmetric functions in infinitely many variables $x_1, x_2, \ldots$ is the ring*

$$\Lambda(x_1, x_2, \ldots) = \mathbb{C}[x_1, x_2, \ldots]^{S_\infty}$$

of formal power series having bounded degree which are symmetric under all permutations of the variables.

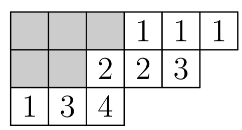
For instance, $x_1^2 + x_2^2 + x_3^2 + \cdots$ is a symmetric function, because interchanging any two of the variables does not change the series.

The most important symmetric functions in this context are the *Schur functions*. They can be defined in many equivalent ways, from being characters of irreducible representations of S_n to an expression as a ratio of determinants. We use the combinatorial definition here, and start by introducing some common terminology involving Young tableaux and partitions.

Definition 12. *A* ***skew shape*** *is the difference λ/μ formed by cutting out the Young diagram of a partition μ from the strictly larger partition λ. A skew shape is a* ***horizontal strip*** *if no column contains more than one box.*

Definition 13. *A* ***semistandard Young tableau (SSYT)*** *of shape ν/λ is a way of filling the boxes of the Young diagram of ν/λ with positive integers so that the numbers are weakly increasing across rows and strictly increasing down columns. An SSYT has* ***content*** *μ if there are μ_i boxes labeled i for each i. The* ***reading word*** *of the tableau is the word formed by concatenating the rows from bottom to top.*

The following is a semistandard young tableau of shape ν/λ and content μ where $\nu = (6, 5, 3)$, $\lambda = (3, 2)$, and $\mu = (4, 2, 2, 1)$. Its reading word is 134223111.



Definition 14. *Let λ be a partition. Given a semistandard Young tableau T of shape λ, define $x^T = x_1^{m_1} x_2^{m_2} \cdots$ where m_i is the number of i's in the tableau T.*

The ***Schur functions*** *are the symmetric functions defined by*

$$s_\lambda = \sum_T x^T$$

where the sum ranges over all SSYT's T of shape λ.

It is well-known that the Schur functions s_λ are symmetric and form a vector space basis of $\Lambda(x_1, x_2, \ldots)$ as λ ranges over all partitions.

Theorem 4. *We have a ring isomorphism*

$$H^*(\mathrm{Gr}(n, k)) \cong \Lambda(x_1, x_2, \ldots)/(s_\lambda | \lambda \not\subset B)$$

where B is the Important Box. The isomorphism sends the Schubert class σ_λ to the Schur function s_λ.

To prove Theorem 4, note that sending σ_λ to s_λ is an isomorphism of the underlying vector spaces, since on the right hand side we have quotiented by the Schur functions whose partition does not fit inside the Important Box. So, it remains to show that this morphism respects the multiplications in these rings, taking cup product to polynomial multiplication.

An important first step is the **Pieri Rule**. For Schur functions, this tells us how to multiply a one-row shape by any other partition:

$$s_{(r)} \cdot s_\lambda = \sum_{\nu/\mu \text{ horz. strip of size } r} s_\nu.$$

We wish to show that the same relation holds for the σ_λ's, that is, that

$$\sigma_{(r)} \cdot \sigma_\lambda = \sum_{\nu/\mu \text{ horz. strip of size } r} \sigma_\nu,$$

where the sum on the right is restricted to partitions ν fitting inside the Important Box.

Note that, by the Duality Theorem, we can multiply both sides of the above relation by σ_{λ^c} to extract the coefficient of σ_λ on the right hand side. So, the Pieri Rule is equivalent to the following restatement:

Theorem 5 (Pieri Rule)**.** *Let λ and μ be partitions with $|\lambda| + |\mu| = k(n-k) - r$. Then if $F_\bullet$, $E_\bullet$, and $H_\bullet$ are three sufficiently general flags then the intersection*

$$\Omega_\lambda(F_\bullet) \cap \Omega_\mu(E_\bullet) \cap \Omega_{(r)}(H_\bullet)$$

has 1 element if μ^c/λ is a horizontal strip, and it is empty otherwise.

Sketch of Proof. We can set $F_\bullet$ and $E_\bullet$ to be the standard and opposite flags and $H_\bullet$ a generic flag, and then perform a direct analysis similar to that in the Duality Theorem. See [2] for full details. □

It turns out that, algebraically, the Pieri rule suffices to show the ring isomorphism, because the Schur functions $s_{(r)}$ (and corresponding Schubert classes $\sigma_{(r)}$ form an algebraic set of generators for their respective rings. Therefore, to intersect Schubert classes we simply have to understand how to multiply Schur functions.

3.6 The Littlewood-Richardson rule

The combinatorial rule for multiplying Schur functions, or Schubert classes, is called the **Littlewood-Richardson Rule**. To state it, we need to introduce a few new notions.

Definition 15. *A word is **Yamanouchi** (or **lattice** or **ballot**) if every suffix (final subword) has at least as many i's as $i+1$'s for all i.*

Definition 16. *A **Littlewood-Richardson tableau** is a semistandard Young tableau whose reading word is Yamanouchi.*

			1	1	1
		2	2	3	
1	3	4			

Exercise 3. The example tableau above is **not** Littlewood-Richardson. Why? Can you find a tableau of that shape that is?

Definition 17. *A sequence of skew tableaux $T_1, T_2, \ldots$ form a **chain** if their shapes do not overlap and*

$$T_1 \cup T_2 \cup \cdots \cup T_i$$

is a partition shape for all i.

We can now state the general Littlewood-Richardson rule. We will refer the reader to [2] for a proof, as the combinatorics is quite involved.

Theorem 6. *We have*

$$s_{\lambda^{(1)}} \cdot \cdots \cdot s_{\lambda^{(m)}} = \sum_{\nu} c^{\nu}_{\lambda^{(1)},\ldots,\lambda^{(m)}} s_{\nu}$$

where $c^{\nu}_{\lambda^{(1)},\ldots,\lambda^{(m)}}$ *is the number of chains of Littlewood-Richardson tableaux of contents* $\lambda^{(i)}$ *with total shape* ν*.*

It follows that in $H^*(\mathrm{Gr}(n,k))$ we have

$$\sigma_{\lambda^{(1)}} \cdot \cdots \cdot \sigma_{\lambda^{(m)}} = \sum_{\nu} c^{\nu}_{\lambda^{(1)},\ldots,\lambda^{(m)}} \sigma_{\nu}$$

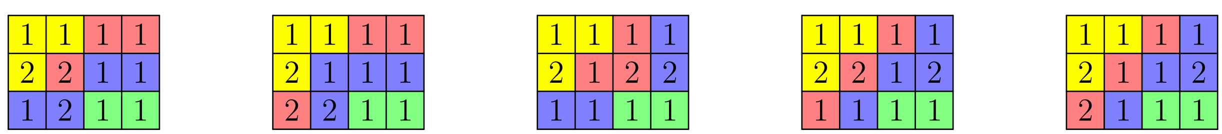
where now the sum on the right is restricted to partitions ν fitting inside the Important Box. Note that by the combinatorics of the Littlewood-Richardson rule, the coefficients on the right are nonzero if and only if $|\nu| = \sum |\lambda^{(i)}|$, and so in the case of a zero-dimensional intersection of Schubert varieties, the only possible ν on the right hand side is the Important Box B itself. Moreover, any element of the cohomology class $\sigma_B = [\Omega_B]$ is a single point of $\mathrm{Gr}(n,k)$, and the zero-dimensional Littlewood-Richardson rule follows as a corollary:

Theorem 7 (Zero-Dimensional Littlewood-Richardson Rule)**.** *Let* B *be the* $k \times (n-k)$ *Important Box, and let* $\lambda^{(1)}, \ldots, \lambda^{(m)}$ *be partitions fitting inside* B *such that* $|B| = \sum_i |\lambda^{(i)}|$*. Also let* $F_\bullet^{(1)}, \ldots, F_\bullet^{(m)}$ *be any* m *generic flags. Then*

$$c^{B}_{\lambda^{(1)},\ldots,\lambda^{(m)}} := |\Omega_{\lambda^{(1)}}(F_\bullet^{(1)}) \cap \cdots \cap \Omega_{\lambda^{(m)}}(F_\bullet^{(m)})|$$

is equal to the number of chains of Littlewood-Richardson tableaux of contents $\lambda^{(1)}, \ldots, \lambda^{(m)}$ *with total shape equal to* B*.*

Example 12. Suppose $k = 3$ and $n - k = 4$. Let $\lambda^{(1)} = (2,1)$, $\lambda^{(2)} = (2,1)$, $\lambda^{(3)} = (3,1)$, and $\lambda^{(4)} = 2$. Then there are four different chains of Littlewood-Richardson tableaux of contents $\lambda^{(1)}, \ldots, \lambda^{(4)}$ that fill the $k \times (n-k)$ Important Box, as shown below.



3.7 Monomial and homogeneous symmetric functions

The material in this section will be helpful for solving problem 0.5 on the Schubert Calculus Quiz.

Define the **monomial symmetric function** m_λ to be the sum of all monomials in $x_1, x_2, \ldots$ having exponents $\lambda_1, \ldots, \lambda_r$. Then it is not hard to see, from the combinatorial definition of Schur functions, that

$$s_\lambda = \sum_{\mu} K_{\lambda\mu} m_\mu$$

where $K_{\lambda\mu}$ is the number of semistandard Young tableaux of shape λ and content μ. The numbers $K_{\lambda\mu}$ are called the **Kostka numbers**, and they can be thought of as a change of basis matrix in the space of symmetric functions.

The **homogeneous symmetric function** h_λ is defined to be $h_{\lambda_1} \cdots h_{\lambda_r}$ where h_d is the sum of *all* monomials of degree d for any given d. The h_λ's also form a $\mathbb{C}$-basis for $\Lambda(x_1, x_2, \ldots)$, and one can then define an inner product on Λ such that

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}.$$

Remarkably, the s_λ's are also orthogonal with respect to this inner product:

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu},$$

and so we have

$$\langle h_\mu, s_\lambda \rangle = \langle h_\mu, \sum_\nu K_{\lambda\nu} m_\nu \rangle = \sum_\nu K_{\lambda\nu} \langle h_\mu, m_\nu \rangle = \sum_\nu K_{\lambda\nu} \delta\mu\nu = K_{\lambda\mu}$$

Thus we have the dual formula

$$h_\mu = \sum_\lambda K_{\lambda\mu} s_\mu.$$

Definition 18. *Define the **hook length** hook(s) of a square s in a Young diagram to be the number of squares strictly to the right of it in its row plus the number of squares strictly below in its column plus 1 for the square itself.*

Theorem 8. *(Hook length formula.) The number of standard Young tableaux of shape λ is*

$$\frac{|\lambda|!}{\prod_{s\in\lambda} \mathrm{hook}(s)}.$$

3.8 Problems

3.1. **Prove Lemma 2:** For any transverse flags $F'_\bullet$ and $E'_\bullet$, there is some $g \in \mathrm{GL}_n$ such that $gF'_\bullet = F_\bullet$ and $gE'_\bullet = E_\bullet$, where $F_\bullet$ and $E_\bullet$ are the standard and opposite flags.

3.2. **It's all Littlewood-Richardson:** Verify that the Duality Theorem and the Pieri Rule are both special cases of the Littlewood-Richardson rule.

3.3. **An empty intersection:** Show that

$$\Omega_{(1,1)}(F_\bullet) \cap \Omega_{(2)}(E_\bullet) = \emptyset$$

in $\mathrm{Gr}(4, 2)$ for transverse flags $F_\bullet$ and $E_\bullet$. What does this mean geometrically?

3.4. **A nonempty intersection:** Show that

$$\Omega_{(1,1)}(F_\bullet) \cdot \Omega_{(2)}(E_\bullet)$$

is nonempty in $\mathrm{Gr}(5, 2)$. (Hint: intersecting it with a certain third Schubert variety will be nonempty by the Littlewood-Richardson rule.) What does this mean geometrically?

3.5. **Problem 2.8 revisited:** In $\mathbb{P}^4$, suppose the 2-planes A and B intersect in a point X, and P and Q are distinct points different from X. Show that there is exactly one plane C that contains both P and Q and intersect A and B each in a line as an intersection of Schubert varieties in $\mathrm{Gr}(5, 3)$, in each of the following cases:

(a) When P is contained in A and Q is contained in B;

(b) When neither P nor Q lie on A or B.

3.6. **There are two lines passing through four given lines in $\mathbb{P}^3$:** Solve problem 0.4 on the Schubert Calculus Quiz for a generic choice of flags.

3.7. **That's a lot of k-planes:** Solve problem 0.5 on the Schubert Calculus Quiz for a generic choice of flags:

(a) Verify that the problem boils down to computing the coefficient of $s_{((n-k)^k)}$ in the product of Schur functions $s_{(1)}^{k(n-k)}$.

(b) Notice that $s_{(1)} = m_{(1)} = h_{(1)} = x_1 + x_2 + x_3 + \cdots$. Thus $s_{(1)}^{k(n-k)} = h_1^{k(n-k)} = h_{(1^{k(n-k)})}$. Show that the coefficient of $s_{((n-k)^k)}$ in $h_{(1^{k(n-k)})}$ is the number of tableaux of shape B with content $(1^{k(n-k)})$, i.e., the number of **standard** Young tableaux of Important Box shape.

(c) Use the Hook Length Formula to finish the computation.

4 Variation 2: The flag variety

The (complete) **flag variety** (in dimension n) is the set of all complete flags in $\mathbb{C}^n$, with a Schubert cell decomposition similar to that of the Grassmannian.

In particular, given a flag $\{V_i\}_{i=1}^n$, we can choose n vectors $v_1, \ldots, v_n$ such that the span of $v_1, \ldots, v_i$ is V_i for each i, and list the vectors v_i as row vectors of an $n \times n$ matrix. We can then perform certain row reduction operations to form a different basis $v'_1, \ldots, v'_n$ that also span the subspaces of the flag, but whose matrix is in the following canonical form: it has 1's in a permutation matrix shape, 0's to the left and below each 1, and arbitrary complex numbers in all other entries.

For instance, say we start with the flag in three dimensions generated by the vectors $\langle 0, 2, 3\rangle$, $\langle 1, 1, 4\rangle$, and $\langle 1, 2, -3\rangle$. The corresponding matrix is

$$\begin{pmatrix} 0 & 2 & 3 \\ 1 & 1 & 4 \\ 1 & 2 & -3 \end{pmatrix}.$$

We start by finding the leftmost nonzero element in the first row and scale that row so that this element is 1. Then subtract multiples of this row from the rows below it so that all the entries below that 1 are 0. Continue the process on all further rows:

$$\begin{pmatrix} 0 & 2 & 3 \\ 1 & 1 & 4 \\ 1 & 2 & -3 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 1.5 \\ 1 & 0 & 2.5 \\ 1 & 0 & -6 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 1.5 \\ 1 & 0 & 2.5 \\ 0 & 0 & 1 \end{pmatrix}$$

It is easy to see that this process does not change the flag formed by the partial row spans, and that any two matrices in canonical form define different flags. So, the flag variety is a cell complex consisting of $n!$ **Schubert cells** given by choosing a permutation and forming the corresponding canonical form. For instance, one such open set in the 5-dimensional flag variety is the open set given by all matrices of the form

$$\begin{pmatrix} 0 & 1 & * & * & * \\ 1 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

We call this cell X°_{45132} because $4, 5, 1, 3, 2$ are the positions of the 1's from the right hand side of the matrix in order from top to bottom. More rigorously:

Definition 19. *Let $w \in S_n$ be a permutation of $\{1, \ldots, n\}$. Then*

$$X^\circ_w = \{V_\bullet \in \mathrm{Fl}_n : \dim(V_p \cap F_q) = \#\{i \le p : w(i) \le q\} \text{ for all } p, q\}$$

where $F_\bullet$ is the standard flag generated by the unit vectors e_{n-i}. (In the matrix form above, the columns are ordered from right to left as before.)

Note that, as in the case of the Grassmannian, we can choose a different flag $F_\bullet$ with respect to which we define our Schubert cell decomposition, and we define $X^\circ_w(F_\bullet)$ accordingly.

The dimension of a Schubert cell X_w is the number of $*$'s in its matrix. The maximum number of $*$'s occurs when the permutation is $w_0 = n(n-1)\cdots 321$, in which case the dimension of the open set X_{w_0} is $n(n-1)/2$ (or $n(n-1)$ over $\mathbb{R}$). In general, it is not hard to see that the number of $*$'s in the set X_w is the **inversion number** $\mathrm{inv}(w)$. This is defined to be the number of pairs of entries $(w(i), w(j))$ of w which are out of order, that is, $i < j$ but $w(i) > w(j)$. Thus we have

$$\dim(X_w^\circ) = \mathrm{inv}(w).$$

Example 13. The permutation $w = 45132$ has seven inversions. (Can you find them all?) We also see that $\dim(X_w^\circ) = 7$, since there are seven $*$ entries in the matrix.

Another useful way to think of $\mathrm{inv}(w)$ is in terms of its **reflection length**.

Definition 20. *Define $s_1, \ldots, s_{n-1} \in S_n$ to be the adjacent transpositions in the symmetric group, that is, s_i is the permutation interchanging i and $i+1$. Then the **reflection length** of w, written $l(w)$, is the smallest number k for which there exists a decomposition*

$$w = s_{i_1} \cdots s_{i_k}.$$

Lemma 3. *We have $l(w) = \mathrm{inv}(w)$ for any $w \in S_n$.*

We will leave the proof of this lemma as an exercise to the reader in the Problems section.

4.1 Variety and cell complex structure

In order to form the attaching maps of our cell complex, we first need to consider the topology induced by viewing the flag variety as a projective variety. In particular, we can use the Plücker embeddings $\mathrm{Gr}(n,k) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$ for each k to embed Fl_n into the larger projective space $\mathbb{P}^{2^n-1}$ whose entries correspond to the Plücker coordinates of each of the initial $k \times n$ submatrices of a given element of the flag variety. This turns out to give a projective subvariety of $\mathbb{P}^{2^n-1}$, which in turn gives rise to a topology on Fl_n.

Now, consider the closures of the sets X_w° in this topology.

Definition 21. *The **Schubert variety** corresponding to a permutation $w \in S_n$ is*

$$X_w = \overline{X_w^\circ}.$$

As in the Grassmannian, these Schubert varieties turn out to be disjoint unions of Schubert cells. The partial ordering in which $X_w = \sqcup_{v \le w} X_v^\circ$ is called the **Bruhat order**, a well-known partial ordering on permutations. (We will briefly review it here, but we refer to [10] for an excellent introduction to Bruhat order.)

Definition 22. *The **Bruhat order** $\le$ on S_n is defined by $v \le w$ if and only if, for every representation of w as a product of $l(w)$ transpositions s_i, one can remove $l(w) - l(v)$ of the transpositions to obtain a representation of v.*

Example 14. The permutation $w = 45132$ can be written as $s_2s_3s_2s_1s_4s_3s_2$. This contains $s_3s_2s_3 = 14325$ as a (non-consecutive) subword, and so $14325 \le 45132$.

4.2 Intersections and Duality

Now suppose we wish to answer incidence questions about our flags: which flags satisfy certain linear constraints? As in the case of the Grassmannian, this boils down to understanding how the Schubert varieties X_w intersect.

We start with the Duality Theorem for Fl_n. Following [2], it will be convenient to define dual Schubert varieties as follows.

Definition 23. *Let $E_\bullet$ be the standard and opposite flags, and for shorthand we let $X_w = X_w(F_\bullet)$ and*

$$Y_w = X_{w_0 \cdot w}(E_\bullet)$$

where $w_0 = n(n-1)\cdots 1$ is the longest word. (Note: Y_w is called a ***dual Schubert variety****, and is written Ω_w in [2].)*

Notice that

$$\dim(Y_w) = \mathrm{inv}(w_0 \cdot w) = n(n-1)/2 - \mathrm{inv}(w)$$

since if $w' = w_0 \cdot w$ then $w'(i) = n + 1 - w(i)$ for all i.

Theorem 9 (Duality Theorem, V2.)**.** *If $l(w) = l(v)$, we have $X_w \cap Y_v = \emptyset$ if $w \neq v$ and $|X_w \cap Y_v| = 1$ if $w = v$. Furthermore, if $l(w) < l(v)$ then $X_w \cap Y_v = \emptyset$.*

The proof works similarly to the Duality Theorem in the Grassmannian. In particular, with respect to the standard basis, the dual Schubert variety Y_w is formed by the same permutation matrix of 1's as in X_w, but with the 0 entries below and to the *right* of the 1's (and $*$ entries elsewhere). For instance, X_{45132} and Y_{45132} look like:

$$\begin{pmatrix} 0 & 1 & * & * & * \\ 1 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} * & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and their intersection contains only the permutation matrix determined by w.

4.3 Schubert polynomials and the cohomology ring

In order to continue our variation on the theme, it would be natural at this point to look for a Pieri rule or a Littlewood-Richardson rule. But just as the cohomology ring of the Grassmannian and the Schur functions made those rules more natural, we now turn to **Schubert polynomials** and the cohomology ring $H^*(\mathrm{Fl}_n)$ over $\mathbb{Z}$.

This ring turns out to be the **coinvariant ring** for the action of S_n on $\mathbb{Z}[x_1, \ldots, x_n]$. Letting σ_w be the cohomology class of Y_w, we have $\sigma_w \in H^{2i}(\mathrm{Fl}_n)$ where $i = \mathrm{inv}(w)$. In particular, for the transpositions s_i, we have $\sigma_{s_i} \in H^2(\mathrm{Fl}_n)$. It turns out that setting $x_i = \sigma_i - \sigma_{i+1}$ for $i \leq n-1$ and $x_n = -\sigma_{s_{n-1}}$ gives a set of generators for the cohomology ring, and in fact

$$H^*(\mathrm{Fl}_n) = \mathbb{Z}[x_1, \ldots, x_n]/(e_1, \ldots, e_n) =: R_n$$

where $e_1, \ldots, e_n$ are the elementary symmetric polynomials in $x_1, \ldots, x_n$.

The Schubert polynomials will be generators of R_n whose product corresponds to the intersection of Schubert varieties. To define them, we require a symmetric difference operator.

Definition 24. *For any polynomial $P(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$, we define*

$$\partial_i(P) = \frac{P - s_i(P)}{x_i - x_{i+1}}$$

where $s_i(P) = P(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n)$ is the polynomial formed by switching x_i and x_{i+1} in P.

We can use these operators to recursively define the Schubert polynomials.

Definition 25. *We define the Schubert polynomials $\mathfrak{S}_w$ for $w \in S_n$ by:*

- $\mathfrak{S}_{w_0} = x_1^{n-1}x_2^{n-2}\cdots x_{n-2}^2x_{n-1}$ *where $w_0 = n(n-1)\cdots 21$ is the longest permutation,*
- *If $w = w_0 \cdot s_{i_1} \cdot \cdots \cdot s_{i_r}$ is a minimal factorization of its form, i.e., $l(w_0 \cdot s_{i_1} \cdot \cdots \cdot s_{i_p}) = n - p$ for all $1 \le p \le r$, then*
$$\mathfrak{S}_w = \partial_{i_r} \circ \partial_{i_{r-1}} \circ \cdots \circ \partial_{i_1}(\mathfrak{S}_{w_0})$$

It turns out that the Schubert polynomials' image in R_n not only form a basis of these cohomology rings, but the polynomials themselves form a basis of all polynomials in the following sense. The Schubert polynomials $\mathfrak{S}_w$ are well-defined for permutations $w \in S_\infty = \bigcup S_m$ for which $w(i) > w(i+1)$ for all $i \ge k$ for some k. For a fixed such k, these Schubert polynomials form a basis for $\mathbb{Z}[x_1, \ldots, x_k]$.

The closest analog of the Pieri rule for Schubert polynomials is known as **Monk's rule**.

Theorem 10 (Monk's rule)**.** *We have*

$$\mathfrak{S}_{s_i} \cdot \mathfrak{S}_w = \sum \mathfrak{S}_v$$

where the sum ranges over all permutations v obtained from w by:

- *Choosing a pair p, q of indices with $p \le i < q$ for which $w(p) < w(q)$ and for any k between p and q, $w(k)$ is not between $w(p)$ and $w(q)$,*
- *Defining $v(p) = w(q)$, $v(q) = w(p)$ and for all other k, $v(k) = w(k)$.*

Equivalently, $v = w \cdot t$ where t is a transposition (pq) with $p \le i < q$ for which $l(v) = l(w) + 1$.

Interestingly, there is not a known "Littlewood-Richardson rule" that generalizes Monk's rule, and this is an important open problem in Schubert calculus.

Open Problem 1. *Find a combinatorial algorithm for computing the coefficients $c_{u,v}^w$ in the expansion*
$$\mathfrak{S}_u \cdot \mathfrak{S}_v = \sum c_{u,v}^w \, \mathfrak{S}_w,$$
and therefore for computing the intersection of Schubert varieties in Fl_n.

Similar open problems exist for other *partial flag varieties*, defined in the next sections.

4.4 Two Alternative Definitions

There are two other ways of defining the flag manifold that are somewhat less explicit but more generalizable. The group $\mathrm{GL}_n(\mathbb{C})$ acts on the set of flags by left multiplication on its sequence of vectors. Under this action, the stabilizer of the standard flag $F_\bullet$ is the subgroup B consisting of all invertible upper-triangular matrices. Notice that GL_n acts transitively on flags via change-of-basis matrices, and so the stabilizer of any arbitrary flag is simply a conjugation gBg^{-1} of B. We can therefore define the flag variety as the quotient GL_n/B, and define its variety structure accordingly.

Alternatively, we can associate to each coset gB in GL_n/B the subgroup gBg^{-1}, and define the flag variety as the set $\mathcal{B}$ of all subgroups conjugate to B. Since B is its own *normalizer* in G ($gBg^{-1} = B$ iff $g \in B$), these sets $\mathcal{B}$ and GL_n/B are in one-to-one correspondence.

4.5 Generalized flag varieties

The notion of a "flag variety" can be extended in an algebraic way starting from the definition as GL_n / B, to quotients of other matrix groups G by certain subgroups B called **Borel subgroups**.

The subgroup B of invertible upper-triangular matrices is an example of a Borel subgroup of GL_n, that is, a **maximal connected solvable subgroup**.

It is *connected* because it is the product of the torus $(\mathbb{C}^*)^n$ and $\binom{n}{2}$ copies of $\mathbb{C}$. We can also show that it is *solvable*, meaning that its derived series of commutators

$$\begin{aligned} B_0 &:= B, \\ B_1 &:= [B_0, B_0], \\ B_2 &:= [B_1, B_1], \\ &\vdots \end{aligned}$$

terminates. Indeed, $[B, B]$ is the set of all matrices of the form $bcb^{-1}c^{-1}$ for b and c in B. Writing $b = (d_1 + n_1)$ and $c = (d_1 + n_2)$ where d_1 and d_2 are diagonal matrices and n_1 and n_2 strictly upper-triangular, it is not hard to show that $bcb^{-1}c^{-1}$ has all 1's on the diagonal. By a similar argument, one can show that the elements of B_2 have 1's on the diagonal and 0's on the off-diagonal, and B_3 has two off-diagonal rows of 0's, and so on. Thus the derived series is eventually the trivial group.

In fact, a well-known theorem of Lie and Kolchin [5] states that *all* solvable subgroups of GL_n consist of upper triangular matrices in some basis. This implies that B is maximal as well among solvable subgroups. Therefore B is a Borel.

The Lie-Kolchin theorem also implies that all the Borel subgroups in GL_n are of the form gBg^{-1} (and all such groups are Borels). That is, all Borel subgroups are conjugate. It turns out that this is true for any semisimple linear algebraic group G, and additionally, any Borel is its own normalizer. By an argument identical to that in the previous section, it follows that the groups G/B are independent of the choice of borel B (up to isomorphism) and are also isomorphic to the set $\mathcal{B}$ of all Borel subgroups of G as well. Therefore we can think of $\mathcal{B}$ as an algebraic variety by inheriting the structure from G/B for any Borel subgroup B.

Finally, we can define the **flag variety** of a linear algebraic group G to be G/B where B is a Borel subgroup.

4.6 Problems

4.1. **Reflection length equals inversion number:** Show that $l(w) = \mathrm{inv}(w)$ for any $w \in S_n$.

4.2. **Practice makes perfect:** Write out all the Schubert polynomials for permutations in S_3 and S_4.

4.3. **The product rule for Schubert calculus:** Prove that $\partial_i(P \cdot Q) = \partial_i(P) \cdot Q + s_i(P) \cdot \partial_i(Q)$ for any two polynomials P and Q.

4.4. **Symmetric difference acts on R_n:** Use the previous problem to show that the operator ∂_i maps the ideal generated by elementary symmetric polynomials to itself, and hence the operator descends to a map on the quotient R_n.

4.5. **Schubert polynomials as a basis:** Prove that if $w \in S_\infty$ satisfies $w(i) > w(i+1)$ for all $i \geq k$ then $\mathfrak{S}_w \in \mathbb{Z}[x_1, \ldots, x_k]$. Show that they form a basis of the polynomial ring as well.

5 Variation 3: The orthogonal Grassmannian

In the previous section, we saw that we can write the Flag variety as G/B where $G = \mathrm{GL}_n$ and B is a Borel subgroup of G. We can generalize this construction to other matrix groups G, and similarly we can generalize the Grassmannian construction by defining it as a quotient as well.

5.1 The Grassmannian revisited

Recall that a **partial flag** is a sequence $F_{i_1} \subset \cdots \subset F_{i_r}$ of subspaces of $\mathbb{C}^n$ with $\dim(F_{i_j}) = i_j$ for all j. The **type** of the partial flag is the sequence of dimensions $(i_1, \ldots, i_r)$. Notice that a k-dimensional subspace of $\mathbb{C}^n$ can be thought of as a partial flag of type (k).

It turns out that all **partial flag varieties**, the varieties of partial flags of certain degrees, can be defined as a quotient G/P for a **parabolic subgroup** P, namely a closed intermediate subgroup $B \subset P \subset G$. The Grassmannian $\mathrm{Gr}(n,k)$, then, can be thought of as the quotient of GL_n by the parabolic subgroup $S = \mathrm{Stab}(V)$ where V is any fixed k-dimensional subspace of $\mathbb{C}^n$. Similarly, we can start with a different algebraic group, say the special orthogonal group SO_{2n+1}, and quotient by parabolic subgroups to get partial flag varieties of other types.

In particular, the **orthogonal Grassmannian** $\mathrm{OG}(2n+1,k)$ is the quotient SO_{2n+1}/P where P is the stabilizer of a fixed **isotropic** k-dimensional subspace V. The term *isotropic* means that V satisfies $\langle v, w\rangle = 0$ for all $v, w \in V$ with respect to a chosen symmetric bilinear form $\langle , \rangle$.

The isotropic condition, at first glance, seems very unnatural. After all, how could a nonzero subspace possibly be so orthogonal to itself? Well, it is first important to note that we are working over $\mathbb{C}$, not $\mathbb{R}$, and the bilinear form is symmetric, not conjugate-symmetric. So for instance, if we choose a basis of $\mathbb{C}^{2n+1}$ and define the bilinear form to be the usual dot product

$$\langle (a_1, \ldots, a_{2n+1}), (b_1, \ldots, b_{2n+1})\rangle = a_1b_1 + a_2b_2 + \cdots + a_{2n+1}b_{2n+1},$$

then the vector $(3, 5i, 4)$ is orthogonal to itself: $3 \cdot 3 + 5i \cdot 5i + 4 \cdot 4 = 0$.

While the choice of symmetric bilinear form does not change the fundamental geometry of the orthogonal Grassmannian, one choice in particular makes things easier to work with in practice: the "reverse dot product" given by

$$\langle (a_1, \ldots, a_{2n+1}), (b_1, \ldots, b_{2n+1})\rangle = \sum_{i=1}^{2n+1} a_i b_{2n+1-i}.$$

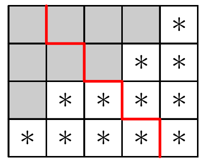
In particular, with respect to this symmetric form, the standard complete flag $F_\bullet$ is an **orthogonal flag**, with $\mathcal{F}_i^\perp = \mathcal{F}_{2n+1-i}$ for all i. Orthogonal flags are precisely the type of flags that are used to define Schubert varieties in the orthogonal grassmannian.

Note that isotropic subspaces are sent to other isotropic subspaces under the action of the orthorgonal group: if $\langle v, w\rangle = 0$ then $\langle Av, Aw\rangle = \langle v, w\rangle = 0$ for any $A \in \mathrm{SO}_{2n+1}$. Thus orthogonal Grassmannian $\mathrm{OG}(2n+1,k)$, which is the quotient $\mathrm{SO}_{2n+1}/\mathrm{Stab}(V)$, can be interpreted as the variety of all k-dimensional isotropic subspaces of $\mathbb{C}^{2n+1}$.

5.2 Schubert varieties and row reduction in $\mathrm{OG}(2n+1,n)$

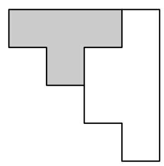
Just as in the ordinary Grassmannian, there is a Schubert cell decomposition for the orthogonal Grassmannian. The combinatorics of the Schubert varieties is particularly nice in the case of $\mathrm{OG}(2n+1,n)$ in which the orthogonal subspaces are "half dimension" n. (In particular, this corresponds to the "cominuscule" Lie type in which the simple root associated to our maximal parabolic subgroup is the special root in type B. See the introduction of [9] or the book [3] for more details.)

In $\mathrm{Gr}(2n+1, n)$, the Schubert varieties are indexed by partitions λ whose Young diagram fit inside the $n \times (n+1)$ Important Box. Suppose we divide this rectangle into two staircases as shown below using the red cut, and only consider the partitions λ that are symmetric with respect to the reflective map taking the upper staircase to the lower.



We claim that the Schubert varieties of the orthogonal Grassmannian are indexed by the **shifted partitions** formed by ignoring the lower half of these symmetric partition diagrams.

Definition 26. *A **shifted partition** is a strictly-decreasing sequence of positive integers, $\lambda = (\lambda_1 > \ldots > \lambda_k)$. We write $|\lambda| = \sum \lambda_i$. The **shifted Young diagram** of λ is the partial grid in which the i-th row contains λ_i boxes and is shifted to the right i steps.*



For instance, the shifted partition above is denoted $(3, 1)$. Notice that the Important Box has turned into an **Important Triangle**, the upper triangle above the staircase cut.

Definition 27. *Let $\mathcal{F}_\bullet$ be an orthogonal flag in $\mathbb{C}^{2n+1}$, and let λ be a shifted partition. Then the **Schubert variety** $X_\lambda(\mathcal{F}_\bullet)$ is defined by*

$$X_\lambda(\mathcal{F}_\bullet) = \{W : \dim(W \cap \mathcal{F}_{n+1+i-\overline{\lambda}_i}) \geq i \text{ for } i = 1, \ldots, n\}$$

where $\overline{\lambda}$ is the "doubled partition" formed by reflecting the shifted partition about the staircase.

In other words, the Schubert varieties consist of the isotropic elements of the ordinary Schubert varieties, giving a natural embedding $\mathrm{OG}(2n+1, n) \to \mathrm{Gr}(2n+1, n)$ that respects the Schubert decompositions:

$$X_\lambda(\mathcal{F}_\bullet) = \Omega_{\overline{\lambda}}(\mathcal{F}_\bullet) \cap \mathrm{OG}(V, n).$$

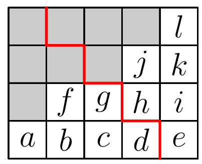
To get a sense of how this works, let's look at the example of $\lambda = (3, 1)$ and $\overline{\lambda} = (4, 3, 1)$ shown above, in the case $n = 4$. The Schubert cell $\Omega^\circ_{\overline{\lambda}}$ in $\mathrm{Gr}(9, 4)$ looks like

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 1 & * & * & 0 & * & 0 & * \\ 1 & * & 0 & * & * & 0 & * & 0 & * \end{bmatrix}$$

Now, which of these spaces are isotropic? Well, suppose we label the starred entries as shown, omitting the 0's:

$$\begin{bmatrix} & & & & & & & 1 & l \\ & & & & & 1 & j & & k \\ & & 1 & f & g & & h & & i \\ 1 & a & & b & c & & d & & e \end{bmatrix} \begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{matrix}$$

It turns out that the entries l, j, k, h, i, e are all uniquely determined by the values of the remaining variables a, b, c, d, f, g. Thus there is one isotropic subspace in this cell for each choice of values a, b, c, d, f, g, corresponding to the "lower half" of the partition diagram we started with:



To see this, let the rows of the matrix be labeled $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ from top to bottom as shown, and suppose its row span is isotropic. Since row $\mathbf{1}$ and $\mathbf{4}$ are orthogonal with respect to the reverse dot product, we get the relation

$$l + a = 0,$$

which expresses $l = -a$ in terms of a.

Now, rows $\mathbf{2}$ and $\mathbf{4}$ are also orthogonal, which means that

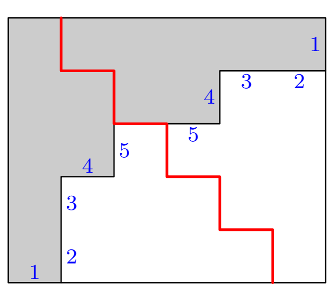
$$b + k = 0,$$

so we can similarly eliminate k. From rows $\mathbf{2}$ and $\mathbf{3}$, we obtain $f + j = 0$, which expresses j in terms of the lower variables. We then pair row $\mathbf{3}$ with itself to see that $h + g^2 = 0$, eliminating h, and finally pairing $\mathbf{3}$ with $\mathbf{4}$ we have $i + gc + d = 0$, so i is now expressed in terms of lower variables as well.

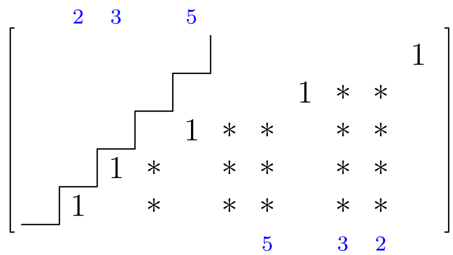
Moreover, these are the only relations we get from the isotropic condition - any other pairings of rows give the trivial relation $0 = 0$. So in this case the Schubert variety restricted to the orthogonal Grassmannian has half the dimension of the original, generated by the possible values for a, b, c, d, f, g.

5.3 General elimination argument

Why does this elimination process work in general, for a symmetric shape λ? Label the steps of the boundary path of λ by $1, 2, 3, \ldots$ from SW to NE in the lower left half, and label them from NE to SW in the upper right half, as shown:



Then the labels on the vertical steps in the lower left half give the column indices of the 1's in the corresponding rows of the matrix. The labels on the horizontal steps in the upper half, which match these labels by symmetry, give the column indices *from the right* of the corresponding starred columns from right to left.



This means that the 1's in the lower left of the matrix correspond to the opposite columns of those containing letters in the upper right half. It follows that we can use the orthogonality relations to pair a 1 (which is leftmost in its row) with a column entry in a higher or equal row so as to express that entry in terms of other letters to its lower left. The 1 is in a lower or equal row in these pairings precisely for the entries whose corresponding square lies above the staircase cut. Thus we can always express the upper right variables in terms of the lower left, as in our example above.

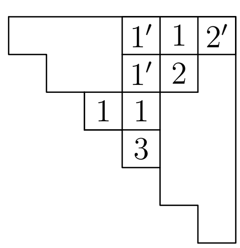
We leave the converse statement, that an isotropic flag comes from a symmetric partition, to the problems section.

5.4 Shifted tableaux and a Littlewood-Richardson rule

The beauty of shifted partitions is that so much of the original tableaux combinatorics that goes into ordinary Schubert calculus works almost the same way for shifted tableaux and the orthogonal Grassmannian. We define these notions rigorously below.

Definition 28. *A* ***shifted semistandard Young tableau*** *is a filling of the boxes of a shifted skew shape with entries from the alphabet* $\{1' < 1 < 2' < 2 < 3' < 3 < \cdots\}$ *such that the entries are weakly increasing down columns and across rows, and such that primed entries can only repeat in columns, and unprimed only in rows.*

The ***reading word*** *of such a tableau is the word formed by concatenating the rows from bottom to top. The* ***content*** *of* T *is the vector* $\mathrm{content}(T) = (n_1, n_2, \ldots)$*, where* n_i *is the total number of* (i)*s and* (i')*s in* T*.*



The tableau above is a shifted semistandard tableau of shape λ/μ where $\lambda = (6,4,2,1)$ and $\mu = (3,2)$, and content $(5,2,1)$. Its reading word is $3111'21'12'$.

In this setting, there are actually two dual types of "Schur functions" that arise from these semistandard tableaux.

Definition 29. *Let λ/μ be a shifted skew shape. Define* $\mathrm{ShST}_Q(\lambda/\mu)$ *to be the set of all shifted semistandard tableaux of shape λ/μ. Define* $\mathrm{ShST}_P(\lambda/\mu)$ *to be the set of those tableaux in which primes are not allowed on the staircase diagonal.*

Definition 30. *The* ***Schur Q-function*** $Q_{\lambda/\mu}$ *is the symmetric function given by*

$$Q_{\lambda/\mu}(X) = \sum_{T \in \mathrm{ShST}_Q(\lambda/\mu)} x^{\mathrm{wt}}(T)$$

and the ***Schur P-function*** $P_{\lambda/\mu}$ *is given by*

$$P_{\lambda/\mu}(X) = \sum_{T \in \mathrm{ShST}_P(\lambda/\mu)} x^{\mathrm{wt}}(T).$$

The Schur Q-functions satisfy the following Littlewood-Richardson-type rule:

$$Q_\mu Q_\nu = \sum 2^{\ell(\mu)+\ell(\nu)-\ell(\lambda)} f^\lambda_{\mu\nu} Q_\lambda$$

It is easy to see that this is equivalent to the rule

$$P_\mu P_\nu = \sum f^\lambda_{\mu\nu} P_\lambda.$$

Here the coefficients $f^\lambda_{\mu\nu}$ are precisely the Littlewood-Richardson coefficients for the orthogonal Grassmannian. In particular, if we extend them to generalized coefficients by

$$P_{\mu^{(1)}} \cdot \dots \cdot P_{\mu^{(r)}} = \sum f^\lambda_{\mu^{(1)} \dots \mu^{(r)}} P_\lambda,$$

then a zero-dimensional intersection $X_{\mu^{(1)}} \cap \dots \cap X_{\mu^{(r)}}$ has exactly $f^T_{\mu^{(1)} \dots \mu^{(r)}}$ points, where T is the Important Triangle.

Stembridge [8] came up with a Littlewood-Richardson-type rule to enumerate these coefficients.

Definition 31. *Let T be a semistandard shifted skew tableau with the first i or i' in reading order unprimed, and with reading word $w = w_1 \cdots w_n$. Let $m_i(j)$ be the multiplicity of i among $w_{n-j+1}, \dots, w_n$ (the last j entries) for any i and for any $j \le n$. Also let $p_i(j)$ be the multiplicity of i' among $w_1, \dots, w_j$. Then T is* ***Littlewood-Richardson*** *if and only if*

- *Whenever $m_i(j) = m_{i+1}(j)$ we have $w_{n-j} \neq i+1, (i+1)'$, and*
- *Whenever $m_i(n) + p_i(j) = m_{i+1}(n) + p_i(j)$ we have $w_{j+1} \neq i, (i+1)'$.*

Notice that this definition implies that $m_i(j) \ge m_{i+1}(j)$ for all i and j, which is the usual Littlewood-Richardson definition for ordinary tableaux.

5.5 Problems

5.1. We can now conversely show that if a point of the Grassmannian is isotropic, then its corresponding partition is symmetric about the staircase cut.

5.2. How many isotropic 3-planes in $\mathbb{C}^7$ intersect six given 3-planes each in at least dimension 1?

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