Computational cutgeneratingfunctionology:

Certifying next-generation cutting planes for mixed integer linear programming

Matthias Köppe

University of California, Davis,
Department of Mathematics

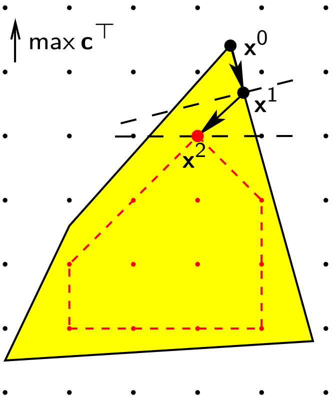
cut·gen·er·at·ing·func·tion·o·lo·gy noun

1. the study of problem-independent functions that map coefficients of constraint systems of mixed integer optimization problems to coefficients of valid inequalities.
2. the study of spaces of such functions.

First known use:

- **cut-generating functions:**
  Conforti, Cornuéjols, Daniilidis, Lemaréchal, Malick, Mathematics of Operations Research, 2015
- **cutgeneratingfunctionologist:**
  Hong, Köppe, Zhou, Proc. ICMS 2016

A very brief history of cutting planes for discrete optimization



- 1954: Dantzig, Fulkerson, Johnson:
  "Large-scale" TSP with LP + cuts
- 1958–1965: Gomory:
  General-purpose cutting planes
- 1970s: Nemhauser, Padberg, Chvátal, Trotter, Balas, Wolsey:
  Polyhedral combinatorics
- 1979–1980s: Grötschel, Padberg, Pulleyblank, Cornuéjols, Naddef:
  Major breakthroughs on TSP using cuts
- 1980s–1990s: . . .
  Polyhedral combinatorics cottage industry
- 1995–: Balas, Ceria, Cornuéjols, Natraj
  Revival of Gomory cuts

⋮

- 2020s: ??
  Next-generation, multi-row, multi-cut cutting plane systems (?)

Gomory's paper on the corner polyhedron

LINEAR ALGEBRA AND ITS APPLICATIONS 451

Some Polyhedra Related to Combinatorial Problems

RALPH E. GOMORY*†
IBM, T. J. Watson Research Center
Yorktown Heights, New York

Communicated by Alan J. Hoffman

ABSTRACT

This paper first describes a theory and algorithms for asymptotic integer programs. Next, a class of polyhedra is introduced. The vertices of these polyhedra provide solutions to the asymptotic integer programming problem; their faces are cutting planes for the general integer programming problem and, to some extent, the polyhedra coincide with the convex hull of the integer points satisfying a linear programming problem. These polyhedra are next shown to be cross sections of more symmetric higher dimensional polyhedra whose properties are then studied. Some algorithms for integer programming, based on a knowledge of the polyhedra, are outlined.

INTRODUCTION

It is well known that a great variety of combinatorial problems can be written as integer programming problems, that is, as systems of inequalities:

$$A'x' \leqslant b, \qquad x' \geqslant 0, \qquad x' \text{ integer}, \tag{1}$$

together with a linear function $c' \cdot x'$ to be maximized. In (1), A' is an $m \times n$ integer matrix, x' an integer n-vector, and b an integer m-vector.

* This work was supported in part by the Office of Naval Research under contract Nonr 3775-(00), NR 047040.

† Part of this paper was written while the author was a visiting member of the Courant Institute of Mathematics, New York University.

Linear Algebra and Its Applications 2(1969), 451–558

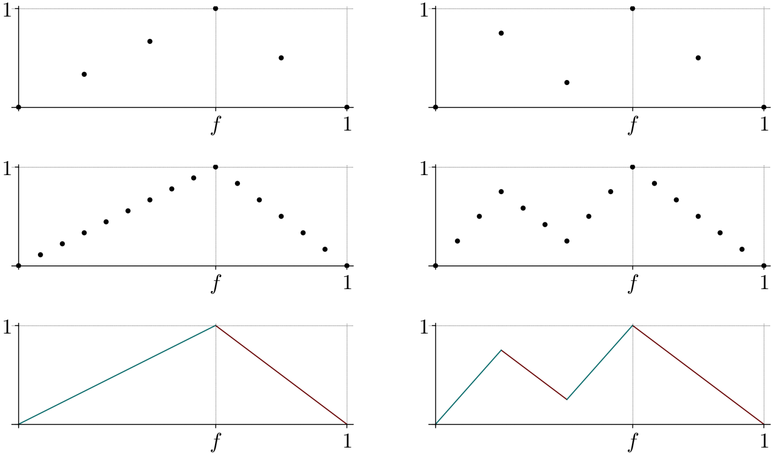
Gomory's group relaxation
($f \in \mathbb{R} \setminus \mathbb{Z}$, $r_j \in \frac{1}{q}\mathbb{Z}$):

$$x = -f + \sum_{j \in N} r_j x_j, \quad x \in \mathbb{Z}, \ \mathbf{x}_N \in \mathbb{Z}_+^N.$$

From Gomory to Gomory–Johnson

Take **Gomory's group relaxation** ($f \in \mathbb{R} \setminus \mathbb{Z}$, $r_j \in \frac{1}{q}\mathbb{Z}$):

$$x = -f + \sum_{j \in N} r_j x_j, \quad x \in \mathbb{Z}, \; \mathbf{x}_N \in \mathbb{Z}_+^N.$$



The Gomory–Johnson papers

Mathematical Programming 3 (1972) 23–85. North-Holland Publishing Company

SOME CONTINUOUS FUNCTIONS RELATED TO CORNER POLYHEDRA

Ralph E. GOMORY and Ellis L. JOHNSON

IBM Research, Yorktown Heights, N.Y., U.S.A.

Received 6 April 1971
Revised manuscript received 16 December 1971

Previous work on Gomory's corner polyhedra is extended to generate valid inequalities for any mixed integer program. The theory of a corresponding asymptotic problem is developed. It is shown how faces previously generated and those given here can be used to give valid inequalities for any integer program.

0. Introduction

0.1.

Inequalities based on the integer nature of some or all of the variables are useful in almost any algorithm for integer programming. They can furnish cut offs for branch and bound or truncated enumeration methods, or cutting planes for cutting plane methods. In this paper we describe methods for producing such inequalities and develop some underlying theory.

We will attempt to outline our general approach, taking the pure integer case first and then the general mixed integer problem.

Consider a pure integer problem

$$Ax = b\,, \quad x \geq 0 \tag{1}$$

in which A is an $m \times (m+n)$ matrix, x is an integer $m+n$ vector, and b an m-vector. If we consider a basis B (in most applications this will be an optimal basis) we can write (1) as

$$Bx_B + Nx_N = b\,, \ x_B \geq 0\,, \ \ x_N \geq 0\,,$$

Mathematical Programming 3 (1972) 359–389. North-Holland Publishing Company

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IBM ...own Heights, N.Y., 10...

... March 1972
...ceived 11 October 1972

The group prob... The connection wit... ties of valid inequalit... between valid inequalit... to give extreme valid ine... to generate such functions. ...c- tions which do not immediate... ...s of faces for certain corner polyhe... ...of the group grows.

...d. with and ...

1. Review of the problems

This paper follows a previous paper [4] but will be self-contained except for proofs of some theorems from [4].

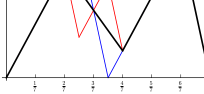
1.1. *The problems* $P(U, u_0)$ *and* $P^+_-(U, u_0)$

Let I be the group formed by the real numbers on the interval [0, 1] with addition modulo 1. Let U be a subset of I and let t be an integer-valued function on U such that (i) $t(u) \geq 0$ for all $u \in U$, and (ii) t has a *finite support*, that is $t(u) > 0$ only for a finite subset U_t of U.

We say that the function t is a *solution to the problem* $P(U, u_0)$, for $u_0 \in I \setminus \{0\}$, if

$$\sum_{u \in U} u\, t(u) = u_0\,. \tag{1.1}$$

Cut-generating functions: valid, minimal, extreme, facet



Take **Gomory's group relaxation** ($f \in \mathbb{R} \setminus \mathbb{Z}$, $r_j \in \mathbb{R}$):

$$x = -f + \sum_{j \in N} r_j x_j, \quad x \in \mathbb{Z}, \ \mathbf{x}_N \in \mathbb{Z}_+^N.$$

Valid inequalities come from **valid (cut-generating) functions** $\pi \colon \mathbb{R} \to \mathbb{R}_+$ in the **Gomory–Johnson model** $R_f(\mathbb{R}/\mathbb{Z})$:

$$\sum_{j \in N} \pi(r_j) x_j \geq 1$$

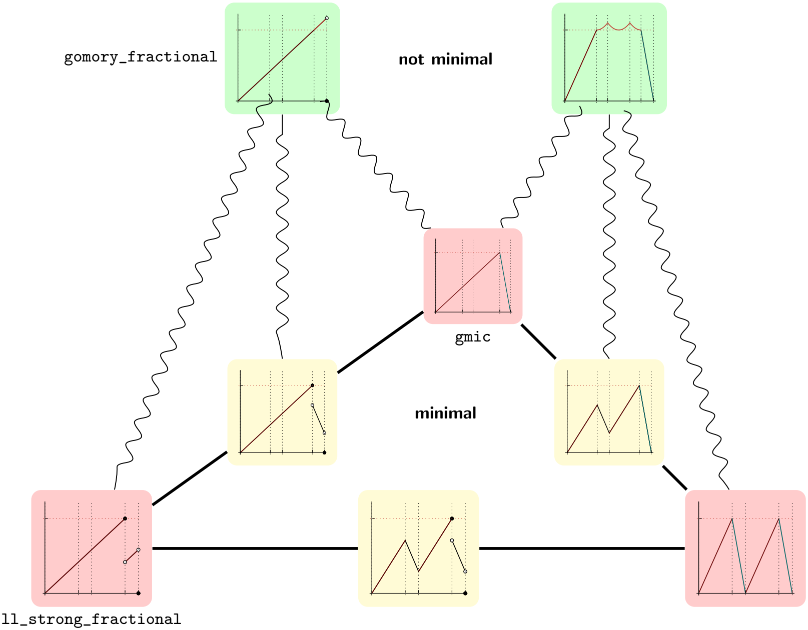
The (pointwise) **minimal valid functions** for $R_f(\mathbb{R}/\mathbb{Z})$ are classified by a theorem by Gomory–Johnson (1972):

1. π is **periodic** modulo 1, $\pi(r) = 0$ for $r \in \mathbb{Z}$,
2. π is **subadditive**: $\Delta\pi(x, y) := \pi(x) + \pi(y) - \pi(x + y) \geq 0$ for $x, y \in \mathbb{R}$,
3. π is **symmetric**: $\pi(x) + \pi(f - x) = 1$ for $x \in \mathbb{R}$.

A minimal function π is **extreme** if it **cannot** be written as a convex combination of two other valid (minimal) functions for $R_f(\mathbb{R}/\mathbb{Z})$:

$$\pi \neq \tfrac{1}{2}\pi_1 + \tfrac{1}{2}\pi_2, \quad \pi \neq \pi_1, \pi_2$$

A hierarchy of functions: valid, minimal, extreme / facet



Recent notable papers in pure cutgeneratingfunctionology

- Conforti, Cornuejols, Daniilidis, Lemarechal, Malick. **Cut-generating functions and S-free sets**, Mathematics of Operations Research, 2015.

$$S \ni \sum_{j \in N} r_j x_j, \quad \mathbf{x}_N \in \mathbb{R}^N_+. \qquad \text{(MLFCB: } S = \mathbb{Z}^k\text{)}$$

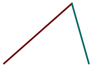
- Yıldız and Cornuejols, **Cut-generating functions for integer variables**, Mathematics of Operations Research, 2016.

$$S \ni \sum_{j \in N} r_j x_j, \quad \mathbf{x}_N \in \mathbb{Z}^N_+. \qquad \text{(Gomory–Johnson: } S = \mathbb{Z}^k\text{)}$$

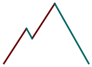
- Basu, Hildebrand, Molinaro, **Minimal cut-generating functions are nearly extreme**, Mathematical Programming, 2018.
- Di Summa, **Piecewise smooth extreme functions are piecewise linear**, Mathematical Programming, 2018.
- Basu, Conforti, Di Summa, Zambelli, **Optimal cutting planes from the group relaxations**, Mathematics of Operations Research, 2019.
- Kılıncc-Karzan, **On Minimal Valid Inequalities for Mixed Integer Conic Programs**. Mathematics of Operations Research, 2016.

An electronic compendium of extreme functions

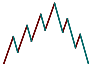
Kö.–Zhou (2014–); available at
`http://mkoeppe.github.io/cutgeneratingfunctionology/doc/html/extreme_functions.html`



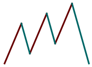
gmic



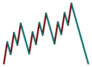
gj_2_slope



gj_2_slope_repeat



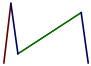
dg_2_step_mir



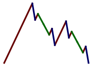
kf_n_step_mir

bccz_counterexample

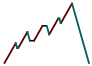
gj_forward_3_slope



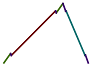
drlm_backward_3_slope



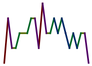
dr_projected_sequential_merge_3_slope



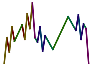
bhk_irrational



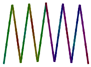
chen_4_slope



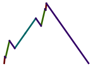
hildebrand_5_slope_22_1



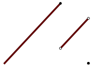
kzh_7_slope_1



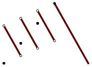
kzh_28_slope_1



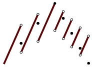
bcdsp_arbitrary_slope



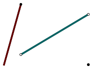
ll_strong_fractional



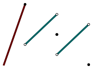
dg_2_step_mir_limit



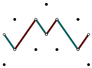
drlm_2_slope_limit



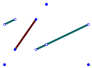
drlm_3_slope_limit



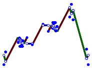
rlm_dpl1_extreme_3a



hildebrand_2_sided_discont_2_slope_1



zhou_two_sided_discontinuous_cannot_assume_any_continuity



kzh_minimal_has_only_crazy_perturbation_1

bcds_discontinuous_everywhere

Cutting plane theorems from the literature

$\pi =$ `drlm_backward_3_slope`

Let $f \in (0,1)$ and $b \in \mathbb{R}$ such that $f < b \leq \frac{1+f}{2}$. Define $\pi\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ as

$$\pi(x) = \begin{cases} \frac{x}{f} & \text{if } 0 \leq x \leq f \\ 1 + \frac{(1+f-b)(x-f)}{(1+f)(f-b)} & \text{if } f \leq x \leq b \\ \frac{x}{1+f} & \text{if } b \leq x \leq 1+f-b \\ \frac{(1+f-b)(x-1)}{(1+f)(f-b)} & \text{if } 1+f-b \leq x \leq 1 \end{cases}$$

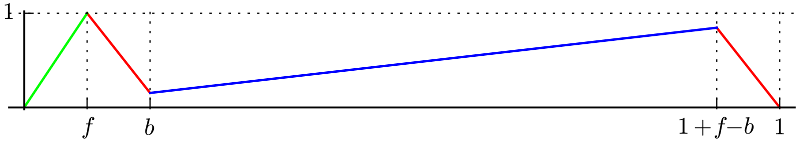


Figure: `drlm_backward_3_slope`

Theorem (Dey–Richard–Li–Miller, 2010; in this form, Köppe–Zhou, 2014)

The function $\pi =$`drlm_backward_3_slope` is an extreme function for $R_f(\mathbb{R}/\mathbb{Z})$, if the parameters satisfy that $0 < f < b \leq \frac{1+f}{4}$.

Cutting plane theorems from the literature

$\pi =$ `chen_4_slope`

Let $f \in (0,1)$, $s^+ > 0, s^- < 0$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.
Define the periodic, piecewise linear function $\pi\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ as:

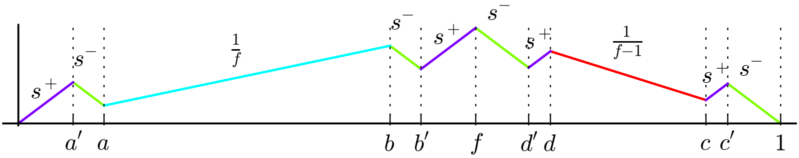


Figure: `chen_4_slope`

where $a' = \frac{\lambda_1(1-s^- f)}{2(s^+-s^-)}$, $a = \frac{\lambda_1 f}{2}$, $c = 1 - \frac{\lambda_2(1-f)}{2}$, $c' = 1 - \frac{\lambda_2(1-s^+(1-f))}{2(s^+-s^-)}$
and $b = f - a$, $b' = f - a'$, $d = 1 + f - c$, $d' = 1 + f - c'$.

Theorem (Chen, 2011, reworded)

The function $\pi =$ `chen_4_slope` is an extreme function for $R_f(\mathbb{R}/\mathbb{Z})$, if the parameters $f, \lambda_1, \lambda_2, s^+$ and s^- satisfy that

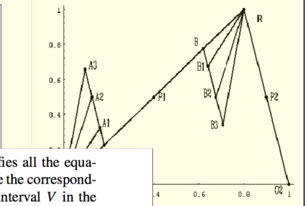
$$f \geq \tfrac{1}{2}, \quad s^+ \geq \tfrac{1}{f}, \quad s^- \leq \tfrac{1}{f-1},$$

$$0 \leq \lambda_1 < \min\{\tfrac{1}{2}, \tfrac{s^+-s^-}{s^+(1-s^- f)}\}, \quad f - \tfrac{1}{s^+} < \lambda_2 < \min\{\tfrac{1}{2}, \tfrac{s^+-s^-}{s^-(s^+(f-1)-1)}\}.$$

Proofs of cutting plane theorems from the literature

$\pi =$ `gj_forward_3_slope`

Theorem 8. *Three Slope Family Theorem: If π is constructed according to Construction 3, and if $\lambda_1 \leq 1/2$ and $0 \leq \lambda_2 \leq 1$, then π is a facet.*



Proof: (A) Minimality and Subadditivity. produced a symmetric π

Subadditivity: Referring to the Subadditivity Checking Theorem, the only convex endpoints in π are the local minima at A and B_1. So the Subadditivity Checking Theorem applied here tells us that we need only check subadditivity for the three following cases.

Case 1: p_1 is A a
Lemma this cann

Case 2: p_1 is B_1
$A) + (B + A) -$
$(B - A) + 2(1 -$
multiple of v_2 so t
inward from $B -$
condition $\lambda \leq 1$ is

Case 3: p_1 is B_1 a
$2v_2 - A - \lambda_2 v_2 =$
Lemma this can n

Proof (B): Uniqueness of the Solution. er any $\pi^*(u)$ that satisfies all the equa- nent $[O, (1/2)w_1]$. Take the correspond- ing interval on G, $u[O, (1/2)w_1]$ as both the interval U and as the interval V in the Interval Lemma. T
$(v, \pi(v))$ on the se
$\pi^*(u) + \pi^*(v) =$
a straight line segr
We next consi
but, we now take [
$u[AA, R]$. Again,
also have $\pi^*(u) +$
the Interval Lemm
Furthermore, from
$U = u[O, (1/2)w$
Continuing to
and $u[R, O2]$, π^*
The slope s_2 i
descends from 1 t
condition, π^* is n
Once s_2 is det
determined by the
tion applies to both
as well. So $\pi^* =$
Since λ, was ar
facets Representat

Because $\lambda_1 \leq \frac{1}{2}$, there is a segment with A as its left end point and $P1 - A$ as the right endpoint. Applying the Interval Lemma to the segments $U = [A, P1 - A]$, $V = [P1 - A, P1]$ and $U + V = [P1, 2P1 - A] = [P1, B]$, we conclude that, since the segments cover $[A, B]$ and π^* is linear with the same slope in each one, and is required to be continuous, π^* must be linear with a single slope over $[A, B]$.

It remains to show that the slope of π^* is the same as the slope of π on $[A, B]$. However $[A, B]$ was constructed on the segment $[O, P1]$ passing through the origin O. Therefore π satisfies both $\pi(2A) = 2\pi(A)$ and $2\pi(P1) = \pi(R)$. These are two relations that π^* must also satisfy. However $\pi^*(2A) = 2\pi^*(A)$ implies that on $u[A, B]$ the linear π^* is part of a line that passes through the origin. In addition $\pi^*(P1) = (\frac{1}{2})\pi^*(R) = \frac{1}{2}$, so π^* passes through $P1$. However, in the vertical strip between O and R and containing $P1$, there is only one line passing through O and $P1$. So π and π^* must have the same slope.

We have now dealt with the segment having the third slope, what remains are the usual segments with slope s^+ and s^-. These are easily dealt with using the Interval Lemma as we did on the discussion of Construction 1.

We now know that the set $E(\pi)$ of all equalities has no solution other than π itself. If we can show that π is subadditive and minimal, we can apply the Facet Theorem.

Producing cutting-plane theorems in quantity

. . . for next-generation, ML/AI-based cutting plane algorithms. . .

The plan:

100,000 postdoctoral positions are available in theoretical, applied and computational optimization for applications in cutgeneratingfunctionology.

A competitive salary based on the applicant's qualifications and experience will be offered.

The positions are available immediately and will be filled as soon as suitable candidates are identified. The candidate should have received (or be about to receive) a doctoral degree in optimization, computational mathematics, operations research, or computer science.

Applicants should send a vita, bibliography with relevant reprints, and a brief description of research experience and interests to

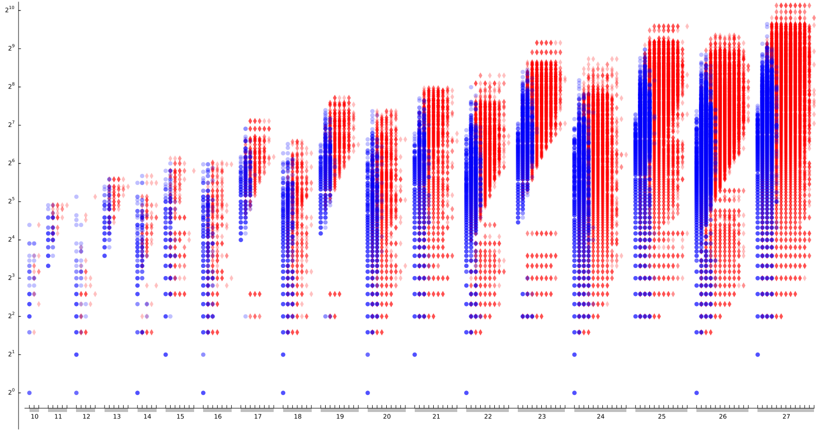
Professor Matthias Koeppe, Department of Mathematics, University of California, Davis, CA 95616 U.S.A. email: mkoeppe@math.ucdavis.edu

Candidates should indicate their availability for massively parallel interviewing at the INFORMS Career Center.

Algorithmic and computational cutgeneratingfunctionology I

Kö.–Zhou, New computer-based search strategies for extreme functions of the Gomory–Johnson infinite group problem, Mathematical Programming Computation, 2017.

Computer-based search for extreme functions: Candidates are interpolations of extreme functions on the $\frac{1}{q}$ grid.



Enabling technology: Polyhedral vertex enumeration

TABLE 1. Efficiency of various vertex enumeration codes without preprocessing

				Running time (s)					
q	dimension	inequalities	vertices	PPL	Porta	cddlib	lrslib	Panda	Normaliz
5	1	21	2	0.001	0.018	0.009	0.008	0.026	0.003
7	2	36	4	0.001	0.012	0.011	0.005	0.026	0.004
9	3	55	7	0.002	0.016	0.018	0.004	0.065	0.005
11	4	78	18	0.003	0.016	0.031	0.009	23	0.007
13	5	105	40	0.007	0.018	0.11	0.021	4604	0.011
15	6	136	68	0.017	0.037	0.21	0.14		0.017
17	7	171	251	0.14	0.20	1.2	0.71		0.047
19	8	210	726	0.91	1.6	5.0	2.3		0.16
21	9	253	1661	6.6	13	24	13		0.67
23[a]	10	300	7188	166	558	785	74		4.9
25	11	351	23214	1854	10048	12129	471		21
26	12	378	54010				2167		62
27	12	406	68216						89
28	13	435	195229						326
29	13	465	317145						644
30	14	496	576696						1693
31	14	528	1216944						3411

Algorithmic and computational cutgeneratingfunctionology II

The Equivariant Perturbation project (2012–):

Given a (piecewise linear) minimal function π, compute its space of effective perturbations:

$$\tilde{\Pi}^{\pi} = \left\{ \tilde{\pi} \colon \mathbb{R}^k \to \mathbb{R} \mid \exists\, \epsilon > 0 \text{ s.t. } \pi^{\pm} = \pi \pm \epsilon\tilde{\pi} \text{ minimal} \right\}.$$

(π is extreme iff $\tilde{\Pi}^{\pi} = \{0\}$.)

- Basu, Hildebrand, **Kö.**, **Equivariant perturbation in Gomory and Johnson's infinite group problem. I. The one-dimensional case.** Mathematics of Operations Research, 2014.

  ⋮

- Hong, **Kö.**, Zhou, **Equivariant perturbation in Gomory and Johnson's infinite group problem. V. Software for the continuous and discontinuous 1-row case.** Optimization Methods and Software, 2018.

  ⋮

- Hildebrand, **Kö.**, Zhou, **Equivariant perturbation in Gomory and Johnson's infinite group problem. VII. Inverse semigroup theory, closures, decomposition of perturbations**, arXiv e-print, 2018

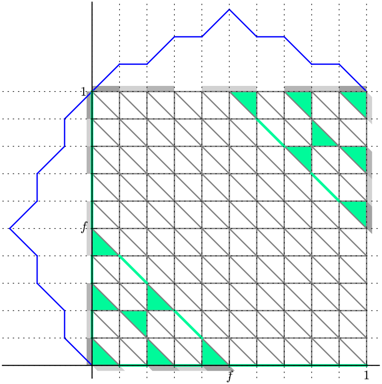
Effective perturbations of minimal functions

Given a minimal function π, what properties does an effective perturbation $\tilde{\pi} \in \tilde{\Pi}^{\pi}$ necessarily have?

For a (possibly discontinuous) piecewise linear function π (on partition $\mathcal{P}$), define a **polyhedral complex** $\Delta\mathcal{P}$ on $\mathbb{R} \times \mathbb{R}$ with faces

$$F(I, J, K) = \{\, (x, y) \in \mathbb{R} \times \mathbb{R} \mid x \in I,\ y \in J,\ x + y \in K \,\}$$

where I, J, K are breakpoints or subintervals of $\mathcal{P}$.



- subadditivity slack

$$\Delta\pi(x, y) = \pi(x) + \pi(y) - \pi(x + y)$$

is affine-linear on $\operatorname{rel\,int}(F)$ for $F \in \Delta\mathcal{P}$.

- Green faces have $\Delta\pi = 0$ on $\operatorname{rel\,int}(F)$
- By convexity, because

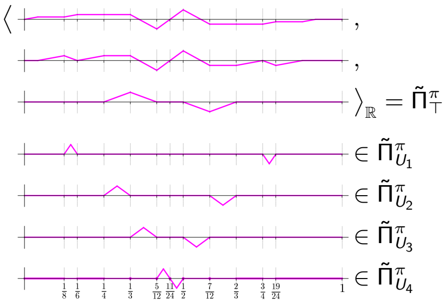
$$\left.\begin{aligned} \pi^{+} &= \pi + \epsilon\tilde{\pi} \\ \pi & \\ \pi^{-} &= \pi - \epsilon\tilde{\pi} \end{aligned}\right\} \text{subadditive,}$$

we have $\Delta\pi(x, y) = 0 \Rightarrow \Delta\tilde{\pi}(x, y) = 0$.

What is ... an equivariant perturbation?

Basu–Hildebrand–**Kö.**: Equivariant Perturbation I; Hildebrand–**Kö.**–Zhou: Equivariant Perturbation VII

$\pi = \texttt{equiv7\_example\_xyz\_2()}$



Group actions – the standard language of symmetries

A group Γ, generated by finitely many translations and point reflections, acts on the space $\mathbb{R}^k$. A function $\tilde{\pi}\colon \mathbb{R}^k \to \mathbb{R}$ is ...

- invariant if:

$$\tilde{\pi}(\gamma * x) = \tilde{\pi}(x)$$

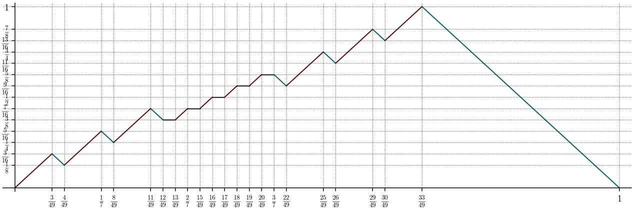
- equivariant if:

$$\tilde{\pi}(\gamma * x) = \gamma \cdot \tilde{\pi}(x)$$

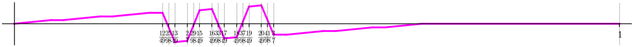
What is . . . an equivariant perturbation (under inverse semigroup actions)?

Hildebrand–**Kö.**–Zhou: Equivariant Perturbation VII

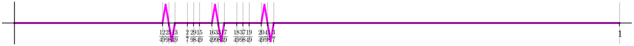
$$\pi_2 = \texttt{equiv7\_minimal\_2\_covered\_2\_uncovered()}.$$

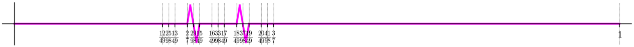


Finite-dimensional perturbation:



Equivariant perturbations:



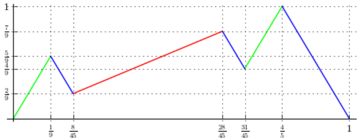


Inverse semigroup actions – to model partial symmetries

Γ an inverse semigroup of **partial maps** (translations and reflections)

Extremality proof by software

Let $\pi = \texttt{gj\_forward\_3\_slope}(f, \lambda_1, \lambda_2)$, with $f = 4/5$, $\lambda_1 = 4/9$ and $\lambda_2 = 2/3$.
The program shows in < 1 second that π is an extreme function.




```
sage: h = gj_forward_3_slope(f=4/5, lambda_1=4/9, lambda_2=2/3)
INFO: 2016-05-15 05:20:40,100 Conditions for extremality are satisfied.
INFO: 2016-05-15 05:20:40,101 Rational case.
sage: extremality_test(h)
                              20:44,157 pi(0) = 0
                              20:44,159 pi is subadditive.
                                ,159 pi is symmetric.
                              20:44,159 Thus pi is minimal.
INFO: 2016-05-15 05:20:44,159 Computing maximal additive faces...
INFO: 2016-05-15 05:20:44,172 Computing maximal additive faces... done
DEBUG: 2016-05-15 05:20:44,179 Step 1: Consider the 2d additive <Face (
[8/45, 4/9], [8/45, 4/9], [16/45, 28/45])>.
[<Int(8/45, 28/45)>] is directly covered.
DEBUG: 2016-05-15 05:20:44,182 Step 2: Consider the 2d additive <Face (
[0, 1/9], [31/45, 4/5], [31/45, 4/5])>.
[<Int(0, 1/9)>, <Int(31/45, 4/5)>] is directly covered.
DEBUG: 2016-05-15 05:20:44,185 Step 3: Consider the 2d additive <Face (
[1/9, 8/45], [28/45, 31/45], [4/5, 13/15])>.
[<Int(1/9, 8/45)>, <Int(28/45, 31/45)>, <Int(4/5, 13/15)>] is directly
covered.
DEBUG: 2016-05-15 05:20:44,191 Step 4: Consider the 2d additive <Face (
[4/5, 1], [4/5, 1], [9/5, 2])>.
[<Int(4/5, 1)>] is directly covered.
DEBUG: 2016-05-15 05:20:44,191 We obtain a  w covered component [<Int(
1/9, 8/45)>, <Int(28/45, 31/45)>, <Int(4/5, )>], with overlapping comp
onents merged in.
INFO: 2016-05-15 05:20:44,                                     moves an
d 3 covered components...
INFO: 2016-05-15 05:20:44,                                    rected mo
ves and 3 covered componen
INFO: 2016-05-15 05:20:44,204 All intervals are covered (or connected-t
o-covered). 3 components.
```

polyhedral complex

functional equations, inverse semigroups

```
DEBUG: 2016-05-15 05:20:44,204 The covered components are [[<Int(8/45,
28/45)>], [<Int(0, 1/9)>, <Int(31/45, 4/5)>], [<Int(1/9, 8/45)>, <Int(2
8/45, 31/45)>, <Int(4/5, 1)>]].
DEBUG: 2016-05-15 05:20:44,316 Let v in R^3.
The i-th entry of v represents the slope parameter on the i-th componen
t of [[<Int(8/45, 28/45)>], [<Int(0, 1/9)>, <Int(31/45, 4/5)>], [<Int(1
/9, 8/45)>, <Int(28/45, 31/45)>, <Int(4/5, 1)>]].
Set up the symbolic function sym: [0,1] -> R^3, so that pert(x) = sym(x
) * v.
The symbolic function sym is <FastPiecewise with 6 parts,
 (0, 1/9)       <FastLinearFunction ((0,1,0))*x>
 (1/9, 8/45)    <FastLine                          -1/9))>
 (8/45, 28/45)  <FastLine                          1/9,-1/15))>
 (28/45, 31/45) <Fas                               9,-5/9))>
 (31/45, 4/5)   <FastLine                          6/45,2/15))>
 (4/5, 1)       <FastLinearFunction ((0,0,1))*x + ((4/9,2/9,-2/3))>>.
DEBUG: 2016-05-15 05:20:44,318 Condition pert(f) = 0 gives the equation
(4/9, 2/9, 2/15) * v = 0.
DEBUG: 2016-05-15 05:20:44,318 Condition pert(1) = 0 gives the equation
(4/9, 2/9, 1/3) * v = 0.
DEBUG: 2016-05-15 05:20:44,319 Condition pert(8/45) + pert(4/9) = pert(
28/45) gives the equation
(-8/45, 1/9, 1/15) * v = 0.
DEBUG: 2016-05-15 05:20:44,323 Solve the li                ations:
[  4/9    2/9  2/15]
[  4/9    2/9   1/3]
[-8/45    1/9  1/15] * v = 0.
INFO: 2016-05-15 05:20:44,333 Finite dimensional test: Solution space h
as dimension 0.
INFO: 2016-05-15 05:20:44,333 Thus the function is extreme.
True
sage:
```

piecewise linear perturbation ansatz

linear algebra

Next: Extend this to parametric families

Algorithmic and computational cutgeneratingfunctionology III

Transform automatic extremality test to a theorem discovery and proof technique

. . . by metaprogramming

Instead of explaining extremality test. . . an easier example:

Anti-definition

A matrix A is **positive definite** if . . . ??? . . .

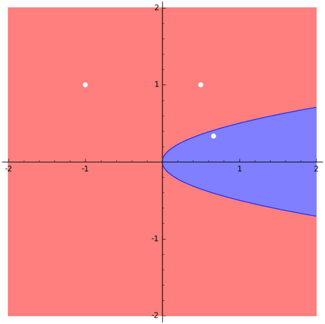
Metaprogramming analysis of a grey-box program

When is the matrix $A = \begin{bmatrix} x & y \\ y & \frac{1}{4} \end{bmatrix}$ positive definite? We have no idea, but we have a trustworthy, real algebraic Python program for testing it for given x and y.

```
A.is_positive_definite()
```

Grey-box model: We can't read the source code to understand what it does, but can apply metaprogramming techniques to analyze the program.

- Testpoint $(x_1, y_1) = (-1, 1)$
  `A.is_positive_definite()` is `False`
  in the proof cell $x < 0$ (side effect).
- Testpoint $(x_2, y_2) = (\frac{1}{2}, 1)$
  `A.is_positive_definite()` is `False`
  in the proof cell $0 < x < 4y^2$.
- Testpoint $(x_3, y_3) = (\frac{2}{3}, \frac{1}{3})$
  `A.is_positive_definite()` is `True`
  in the proof cell $x > 4y^2$.



Our metaprogramming technique SPAM

Simplified Python code

```
class ParametricRealFieldElement(FieldElement):

    ...

    def _add_(self, other):
        return ParametricRealFieldElement(self._val + other._val,
                                          self._sym + other._sym,
                                          parent=self.parent())
    ...

    def __cmp__(left, right):    # Py2
        result = cmp(left._val, right._val)
        if result == 0:
            left.parent().record_to_eq(left.sym() - right.sym())
        elif result == -1:
            left.parent().record_to_lt(left.sym() - right.sym())
        elif result == 1:
            left.parent().record_to_lt(right.sym() - left.sym())
        return result

    ...
```

Automated proof for `drlm_backward_3_slope`

by SPAM* = SPAM + wall crossing

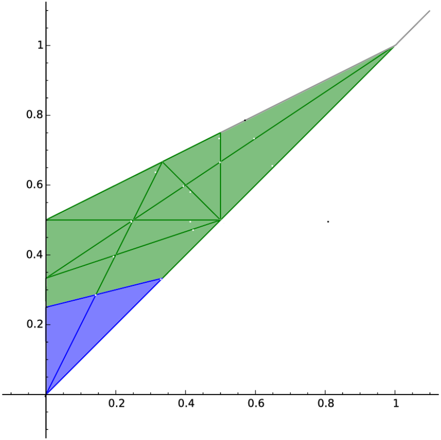


Figure: The cell complexes of `drlm_backward_3_slope`, showing the parameters $(f, bkpt)$.
Cell colors: constructible, but not minimal, minimal, but not extreme, extreme.

Automated proof for `gj_forward_3_slope`

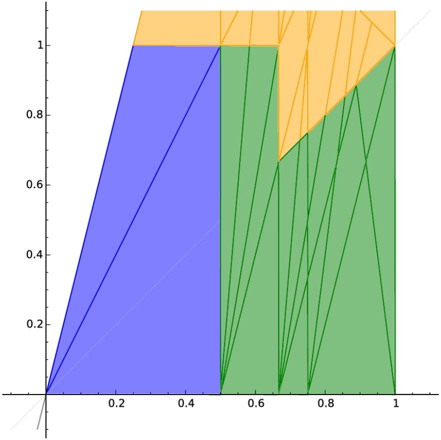


Figure: The cell complex of `gj_forward_3_slope`, showing the parameters (λ_1, λ_2) for fixed $f = 4/5$.

Cell colors: constructible, but not minimal , minimal, but not extreme , extreme .

Automated proof for gj_forward_3_slope

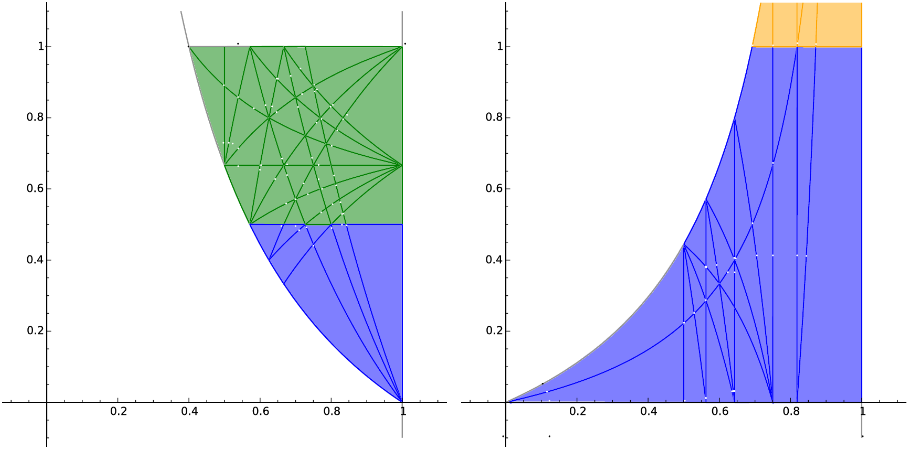


Figure: The cell complex of the parametric family `gj_forward_3_slope`. **Left**, showing the plane of parameters (f, λ_1) for fixed $\lambda_2 = 2/3$; **Right**, showing the plane of parameters (f, λ_2) for fixed $\lambda_1 = 4/9$. Cell colors: constructible, but not minimal, minimal, but not extreme, extreme.

Correction for chen_4_slope

Chen's theorem about his chen_4_slope family is incorrect (conditions for extremality neither necessary nor sufficient).

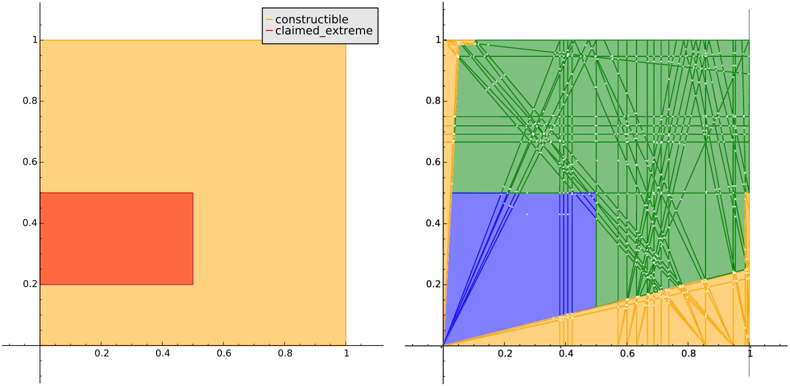


Figure: The extreme region of chen_4_slope claimed in the literature is incorrect. Parameters (λ_1, λ_2); fixed $f = 7/10$, $s^+ = 2$, $s^- = -4$. **Left**, the incorrect hypotheses from Chen's theorem within the region of constructibility. **Right**, the cell complex computed by our implementation. Cell colors: constructible, but not minimal, minimal, but not extreme, extreme.

Correction for `chen_4_slope`

Theorem (Kö., Zhou, & the computer, 2019)

Let $\lambda_1, \lambda_2 \in [0,1)$. Let $\pi\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be the function `chen_4_slope`, where $f = 7/10$, $s^+ = 2$ and $s^- = -4$. Then π is an extreme function for $R_f(\mathbb{R},\mathbb{Z})$, if and only if the parameters λ_1 and λ_2 satisfy that

$$\lambda_1 \le \frac{1}{2}, \quad \lambda_2 \le \frac{1}{2}, \quad (s^+ f - 1)(1 - s^- f)\lambda_1 \le (s^+ - s^-)\lambda_2, \text{ and}$$

$$(s^+(1-f)+1)(s^-(f-1)-1)\lambda_2 \le (s^+ - s^-)\lambda_1.$$

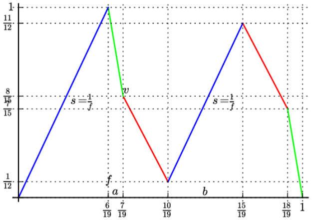
Computer-assisted discovery of new theorems

- Take a function from computer-based search for extreme functions (on a grid)

  Kö., Zhou (2015)

- Invent some parametrization:

```
def param_3_slope_1(f=6/19, a=1/19, b=5/19, v=8/15):
    ...
```



- Run minimality test:

```
sage: K.<f,a,b,v>=ParametricRealField([6/19,1/19,5/19,8/15])
sage: h = param_3_slope_1(f,a,b,v)
sage: minimality_test(h)
True
sage: K._eq_factor
{-f^2*v + 3*f*b*v + f^2 + f*a - 3*f*b - 3*a*b - f*v + b}
```

- New equation! Eliminate v.

Computer-assisted discovery of new theorems

Revise definition:

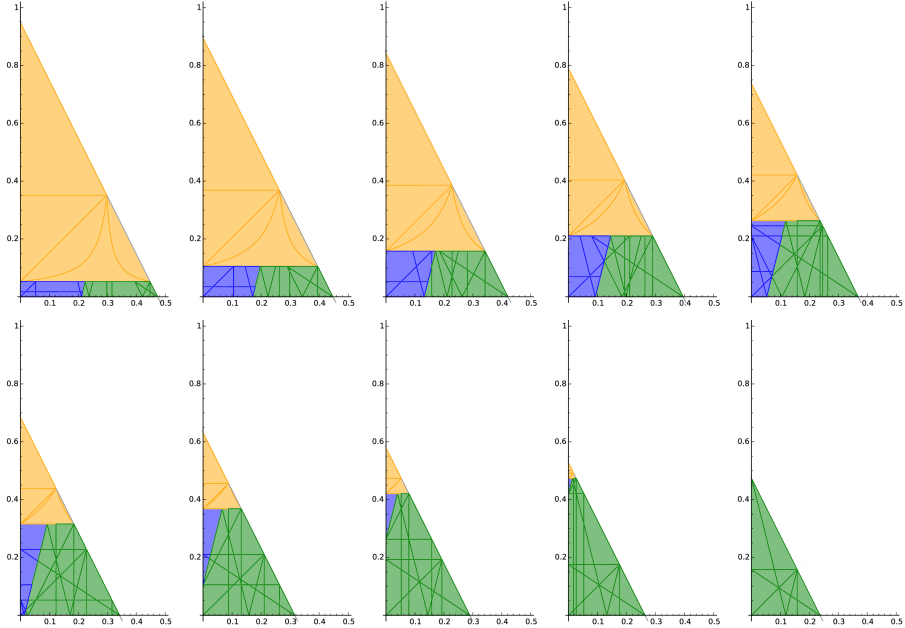
```
def kzh_3_slope_param_extreme_1(f=6/19, a=1/19, b=5/19, field=None,
                                conditioncheck=True):
    ...
    v = (f*f+f*a-3*f*b-3*a*b+b)/(f*f+f-3*f*b)
    bkpts = [0, f, f+a, (1+f-b)/2, (1+f+b)/2, 1-a, 1]
    values = [0, 1, v, (f-b)/2/f, (f+b)/2/f, 1-v, 0]
    return piecewise_function_from_breakpoints_and_values
             (bkpts, values, field=field)
```

Rerun the algorithm — full-dimensional cell

```
3*f + 4*a - b - 1 < 0
-f^2 - f*a + 3*f*b + 3*a*b - b < 0
f*a - 3*a*b - f + b < 0
-f^2*a + 3*f*a*b - 3*a*b - f + b < 0
-a < 0
-f + b < 0
-f - 3*b + 1 < 0
```

Compute the complex by wall-crossing search

Computer-assisted discovery of new theorems



Computer-assisted discovery of new theorems

Theorem (Kö., Zhou, & the computer, ISCO 2016)

Let $f \in (0,1)$ and $a, b \in \mathbb{R}$ such that

$$0 \leq a, \quad 0 \leq b \leq f \text{ and } 3f + 4a - b - 1 \leq 0.$$

The periodic, piecewise linear `kzh_3_slope_param_extreme_1` *function $\pi \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ defined as follows is extreme. The function π has breakpoints at*

$$0, f, f + a, \frac{1+f-b}{2}, \frac{1+f+b}{2}, 1-a, 1.$$

The values at breakpoints are given by $\pi(0) = \pi(1) = 0$, $\pi(f+a) = 1 - \pi(1-a) = v$ and $\pi(\frac{1+f-b}{2}) = 1 - \pi(\frac{1+f+b}{2}) = \frac{f-b}{2f}$, where $v = \frac{f^2+fa-3fb-3ab+b}{f^2+f-3bf}$.

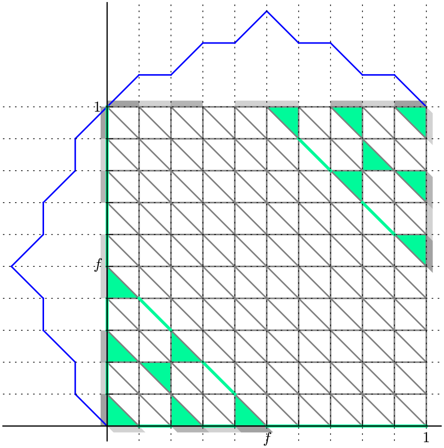
Open: Global structure of the space of extreme functions

`gj_2_slope(f, λ)`

Revisiting the minimality test

Kö., Wang, 2019

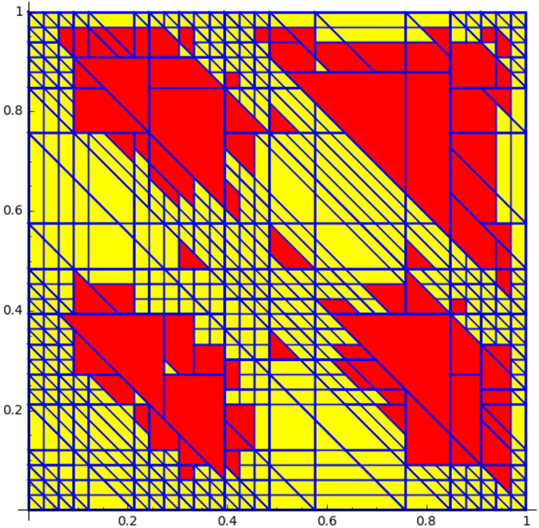
Prove minimality without visiting all vertices of the polyhedral complex $\Delta\mathcal{P}$?



...by spatial branch and bound, minimizing the subadditivity slack $\Delta\pi$...

Example: $\pi = kzh\_7\_slope\_1()$, piecewise constant bounds

White regions: non-leaf
Yellow regions (leaf): Pruned by indivisible regions
Red regions (leaf): Pruned by bounding using estimators



Performance profile

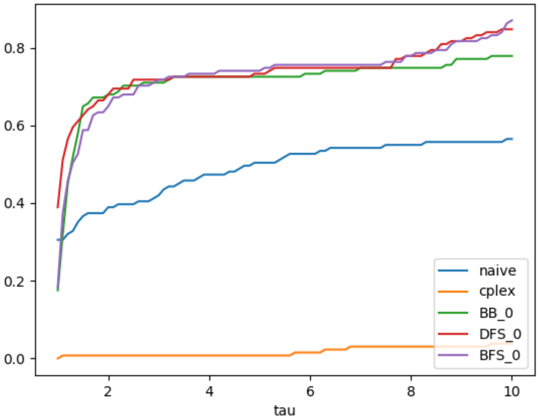


Figure: Performance profiles of 5 algorithms. Three spatial B&B algorithms all use constant bounds with different node selection strategies. The CPLEX approach uses Vielma's disaggregated logarithmic (DLog) formulation, which contains $O(log_2 n)$ binary vairables.

Conclusion

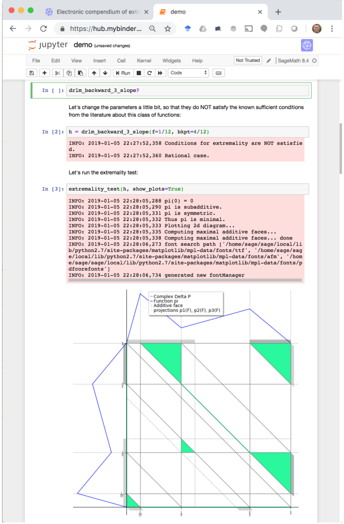
"These new theorems are entirely unremarkable; the plan for the future is to make up for it by sheer quantity."

`https://github.com/mkoeppe/cutgeneratingfunctionology`

SageMath (Python) package cutgeneratingfunctionology

https://github.com/mkoeppe/cutgeneratingfunctionology

Authors: Chun Yu Hong (2013), **Kö.** (2013–), Yuan Zhou (2014–), Jiawei Wang (2016–), contributing undergraduate programmers



Models:

- **1-row Gomory–Johnson model**
- Gomory's finite (cyclic) group problem
- superadditive lifting functions
- classical, general dual-feasible functions
- multi-row code under development

Functionality:

- electronic compendium of functions
- automatic extremality test (Basu–Hildebrand–**Kö.**, Math. Oper. Res. 2014, Hong–**Kö.**–Zhou, ICMS 2016, Hildebrand–**Kö.**–Zhou, 2018+)
- computer-based search for extreme functions (**Kö.**–Zhou, MPC 2016)
- parametric analysis (**Kö.**–Zhou, ISCO 2016)