Mathematics for Decision Making: An Introduction

Lecture 17

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Minimum-cost flow problems

Minimum-cost r-s flow problem

Given a digraph (V, A), source r, sink s, arc capacities $u_{v,w}$, per-unit costs $c_{v,w}$, and a flow value ϕ:
Find a feasible flow $\mathbf{x}$ of value $f_{\mathbf{x}}(s) = \phi$ that has minimum total flow costs $\sum c_{v,w} x_{v,w}$.

We can generalize this to problems with **several sources and sinks**. (Note this is still the case of **one commodity** – i.e., the same kinds of goods are produced in the sources and consumed in the sinks, so it does not matter to which sink something is sent.)

Minimum-cost flow problem

Given a digraph (V, A), arc capacities $u_{v,w}$, and flow excess values b_v for all nodes, find a **feasible flow**, i.e., a vector $\mathbf{x}$ of arc flows $x_{v,w}$ with

$$0 \leq x_{v,w} \leq u_{v,w}$$

and

$$f_{\mathbf{x}}(v) = b_v,$$

that has **minimum total flow costs** $\sum c_{v,w} x_{v,w}$.

The primal criterion of optimality

- By definition, a feasible flow $\mathbf{x}^1$ for the minimum-cost flow problem has minimal cost if and only if there exists no feasible flow $\mathbf{x}^2$ of smaller cost.
- So let's consider a feasible flow $\mathbf{x}^2$ as a candidate.
- Call $\bar{\mathbf{x}} = \mathbf{x}^2 - \mathbf{x}^1$ the difference of the two flows.
- Since both $\mathbf{x}^1$ and $\mathbf{x}^2$ satisfy the equations $f_{\mathbf{x}}(v) = b_v$ for all v, we have

$$f_{\bar{\mathbf{x}}}(v) = 0 \qquad \text{for all } v.$$

- From $\mathbf{0} \leq \mathbf{x}^1 + \bar{\mathbf{x}} \leq \mathbf{u}$, we also have the lower and upper bounds

$$-\mathbf{x}^1 \leq \bar{\mathbf{x}} \leq \mathbf{u} - \mathbf{x}^1.$$

- Finally, $\mathbf{x}^2$ has smaller cost if and only if $\bar{\mathbf{x}}$ has negative cost:

$$\sum_{(v,w)\in A} c_{v,w}\bar{x}_{v,w} < 0$$

- These three conditions **characterize** "difference flows" $\bar{\mathbf{x}}$ that can be added to the feasible flow $\mathbf{x}^1$, to obtain a new feasible flow ($\mathbf{x}^2$) of smaller cost.

Using auxiliary networks

- Components of $\bar{\mathbf{x}}$ can be negative. To work around this, if $x^1_{v,w} > 0$, we write

$$\bar{x}_{v,w} = z_{v,w} - z^{\text{rev}}_{w,v}$$

with non-negative variables that respect the bounds

$$0 \le z_{v,w} \le u_{v,w} - x^1_{v,w}$$
$$0 \le z^{\text{rev}}_{w,v} \le x^1_{v,w}.$$

- **We can interpret this as a feasible flow z (without source or sink, i.e., a circulation) in the auxiliary network $G(\mathbf{x}^1)$.**
- Note that the auxiliary graph does not have arcs corresponding to variables $z_{v,w}$ and $z^{\text{rev}}_{w,v}$ that are fixed to zero by the above bounds.
- (Note that the relation between $\bar{\mathbf{x}}$ and $\mathbf{z}$ is one-to-many.)

Theorem

A feasible flow $\mathbf{x}^1$ has minimal cost if and only if there does not exist a feasible circulation $\mathbf{z}$ in the auxiliary network (with the given capacities) with negative cost

$$\mathbf{c}(\mathbf{z}) := \sum_{a \in A(\mathbf{x}^1)} (c_{v,w} z_{v,w} - c_{v,w} z^{\text{rev}}_{w,v}).$$

Using auxiliary networks

- Now, from the Flow Decomposition Theorem, we know that every circulation can be decomposed into flows along (simple) directed circuits:

$$\mathbf{z} = \sum_{i=1}^{k} \lambda_i \mathbf{z}^i$$

(with $\lambda_i \geq 0$, and $\mathbf{z}^i$ a unit flow along a simple directed circuit, and $k \leq |A|$)

- Since $\mathbf{c}(\mathbf{z}) = \sum_{i=1}^{k} \lambda_i \mathbf{c}(\mathbf{z}^i)$, we know that if $\mathbf{c}(\mathbf{z}) < 0$, then at least one $\mathbf{c}(\mathbf{z}^i) < 0$, so there exists a simple directed circuit of negative cost in $G(\mathbf{x}^1)$.
- On the other hand, if $\mathbf{z}^i$ is a (unit) flow along a simple directed circuit in $G(\mathbf{x}^1)$ with $\mathbf{c}(\mathbf{z}^i) < 0$, then $\mathbf{x}^1$ is not minimal (because we can augment $\mathbf{x}^1$ by sending some $\lambda_i > 0$ units of flow along the circuit, which will decrease the total cost).

Theorem

A feasible flow $\mathbf{x}^1$ has minimal cost if and only if there does not exist a simple directed circuit of negative cost in the auxiliary network.

Augmenting Circuit Algorithm for Min Cost Flow

Augmenting Circuit Algorithm, Kantoróvich [1942]

Input: Graph $G = (V, A)$, capacities $\mathbf{u}$, excess values $\mathbf{b}$, costs $\mathbf{c}$

- Find a feasible flow $\mathbf{x}$ (max-flow, homework)
- While there exists a negative-cost directed circuit in $G(\mathbf{x})$, i.e., an **augmenting circuit** for $\mathbf{x}$ in G:
  Determine the capacity (bottleneck) of the augmenting circuit.
  Augment $\mathbf{x}$ along C by this bottleneck.

- Negative-cost directed circuits can be determined (in polynomial time) by running the Bellman–Ford algorithm, or other shortest-path algorithms. (Key: cycles in the predecessor vector.)
- Again, as we see already on simple examples, this gives us (only) a **pseudo-polynomial algorithm** for instances where it terminates.
- Choosing "most negative" augmenting circuits does not work (neither effective nor efficient)
- Choosing minimum-mean-cost (i.e., most-negative-mean-cost) circuits produces a polynomial-time algorithm, Goldberg–Tarjan [1989]
- A strongly polynomial algorithm for min-cost flow was unknown until 1985!