Quiz 2: Solutions

1. Consider the difference equation:

\[ a_{k+1} = \frac{1}{2} a_k + \frac{5}{4}, \text{ where } a_0 = 1, k = 0, 1, 2 \ldots \]

(a) Compute \(a_1, a_2, a_3, a_4\).

(b) As \(n\) increases, what does \(a_n\) seem to approach?

Solution:

\[
\begin{align*}
a_1 &= \frac{1}{2} a_0 + \frac{5}{4} = \frac{1}{2}(1) + \frac{5}{4} = \frac{7}{4} = 1.75 \\
a_2 &= \frac{1}{2} a_1 + \frac{5}{4} = \frac{1}{2} \left( \frac{7}{8} \right) + \frac{5}{4} = \frac{17}{8} = 2.125 \\
a_3 &= \frac{1}{2} a_2 + \frac{5}{4} = \frac{1}{2} \left( \frac{17}{16} \right) + \frac{5}{4} = \frac{37}{16} = 2.3125 \\
a_4 &= \frac{1}{2} a_3 + \frac{5}{4} = \frac{1}{2} \left( \frac{37}{32} \right) + \frac{5}{4} = \frac{77}{32} = 2.40625
\end{align*}
\]

As \(n\) increases it seems like \(a_n\) approaches 2.5. (To check this let’s see if \(a_n\) has any fixed points, this is a point \(a\) such that

\[ a = \frac{1}{2} a + \frac{5}{4} \]

\[ \frac{1}{2} a = \frac{5}{4} \]

\[ a = \frac{5}{2} = 2.5 \]

Since \(a_n\) is strictly increasing as \(n\) increases and its values are approaching the fixed point from below, then we know that its limiting behavior should approach the fixed point. Therefore we may be confident in our guess that as \(n\) increases it seems like \(a_n\) approaches 2.5. This justification is correct but was not needed to achieve full credit.)

2. Suppose you were studying a set of data, and a member of your lab told you that when \(\log(y)\) versus \(x\) was plotted, it was well fit by a linear line passing through the points \(A = (0, 3)\) and \(B = (9, 1)\). These are part of a semi-log (log-linear) graph. Graph the line in the log-linear plot, and find the functional relationship \(y = f(x)\). Identify this relationship as either exponential or power function.

Solution:
Above is the log-linear graph of the line passing through points $A = (0, 3)$ and $B = (9, 1)$. To get full credit here, you should have labeled the axes correctly. First find the equation of this line. The slope is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 3}{9 - 0} = -\frac{2}{9}$$

where the $y$-intercept is 3. Therefore the equation of the line is

$$Y = -\frac{2}{9}x + 3.$$ 

Since we used a log-linear transformation, we have that $Y = \log_{10}(y)$ (here we assume that base 10 was used in the log-linear transformation). Therefore

$$\log_{10}(y) = Y = -\frac{2}{9}x + 3$$

$$y = 10^{-\frac{2}{9}x+3}$$

$$= 10^{-\frac{2}{9}x}10^3$$

The above is the functional relationship and it is an exponential function.