6.1 INTEGRATION BY SUBSTITUTION

- Use the basic integration formulas to find indefinite integrals.
- Use substitution to find indefinite integrals.
- Use substitution to evaluate definite integrals.
- Use integration to solve real-life problems.

Review of Basic Integration Formulas

Each of the basic integration rules you studied in Chapter 5 was derived from a corresponding differentiation rule. It may surprise you to learn that, although you now have all the necessary tools for *differentiating* algebraic, exponential, and logarithmic functions, your set of tools for *integrating* these functions is by no means complete. The primary objective of this chapter is to develop several techniques that greatly expand the set of integrals to which the basic integration formulas can be applied.

Basic Integration Formulas

1. Constant Rule: $\int k\,dx = kx + C$

2. Simple Power Rule $(n \neq -1)$: $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

3. General Power Rule $(n \neq -1)$: $\int u^n \frac{du}{dx}\,dx = \int u^n\,du = \frac{u^{n+1}}{n+1} + C$

4. Simple Exponential Rule: $\int e^x\,dx = e^x + C$

5. General Exponential Rule: $\int e^u \frac{du}{dx}\,dx = \int e^u\,du = e^u + C$

6. Simple Log Rule: $\int \frac{1}{x}\,dx = \ln|x| + C$

7. General Log Rule: $\int \frac{du/dx}{u}\,dx = \int \frac{1}{u}\,du = \ln|u| + C$

As you will see once you work a few integration problems, integration is not nearly as straightforward as differentiation. A major part of any integration problem is determining which basic integration formula (or formulas) to use to solve the problem. This requires remembering the basic formulas, familiarity with various procedures for rewriting integrands in the basic forms, and lots of practice.

Integration by Substitution

There are several techniques for rewriting an integral so that it fits one or more of the basic formulas. One of the most powerful techniques is **integration by substitution.** With this technique, you choose part of the integrand to be u and then rewrite the entire integral in terms of u.

EXAMPLE 1 Integration by Substitution

Use the substitution $u = x + 1$ to find the indefinite integral.

$$\int \frac{x}{(x+1)^2}\,dx$$

SOLUTION From the substitution $u = x + 1$,

$$x = u - 1, \quad \frac{du}{dx} = 1, \quad \text{and} \quad dx = du.$$

By replacing *all* instances of x and dx with the appropriate u-variable forms, you obtain

$$\begin{aligned}
\int \frac{x}{(x+1)^2}\,dx &= \int \frac{u-1}{u^2}\,du && \text{Substitute for } x \text{ and } dx.\\
&= \int \left(\frac{u}{u^2} - \frac{1}{u^2}\right) du && \text{Write as separate fractions.}\\
&= \int \left(\frac{1}{u} - \frac{1}{u^2}\right) du && \text{Simplify.}\\
&= \ln|u| + \frac{1}{u} + C && \text{Find antiderivative.}\\
&= \ln|x+1| + \frac{1}{x+1} + C. && \text{Substitute for } u.
\end{aligned}$$

The basic steps for integration by substitution are outlined in the guidelines below.

Guidelines for Integration by Substitution

1. Let u be a function of x (usually part of the integrand).
2. Solve for x and dx in terms of u and du.
3. Convert the entire integral to u-variable form and try to fit it to one or more of the basic integration formulas. If none fits, try a different substitution.
4. After integrating, rewrite the antiderivative as a function of x.

STUDY TIP

When you use integration by substitution, you need to realize that your integral should contain just one variable. For instance, the integrals

$$\int \frac{x}{(x+1)^2}\,dx$$

and

$$\int \frac{u-1}{u^2}\,du$$

are in the correct form, but the integral

$$\int \frac{x}{u^2}\,dx$$

is not.

TRY IT 1

Use the substitution $u = x - 2$ to find the indefinite integral.

$$\int \frac{x}{(x-2)^2}\,dx$$

DISCOVERY

Suppose you were asked to evaluate the integrals below. Which one would you choose? Explain your reasoning.

$$\int \sqrt{x^2+1}\,dx \quad \text{or}$$

$$\int x\sqrt{x^2+1}\,dx$$

EXAMPLE 2 Integration by Substitution

Find $\int x\sqrt{x^2-1}\,dx$.

SOLUTION Consider the substitution $u = x^2 - 1$, which produces $du = 2x\,dx$. To create $2x\,dx$ as part of the integral, multiply and divide by 2.

$$\int x\sqrt{x^2-1}\,dx = \frac{1}{2}\int \underbrace{(x^2-1)^{1/2}}_{u^{1/2}}\underbrace{2x\,dx}_{du} \qquad \text{Multiply and divide by 2.}$$

$$= \frac{1}{2}\int u^{1/2}\,du \qquad \text{Substitute for } x \text{ and } dx.$$

$$= \frac{1}{2}\frac{u^{3/2}}{3/2} + C \qquad \text{Power Rule}$$

$$= \frac{1}{3}u^{3/2} + C \qquad \text{Simplify.}$$

$$= \frac{1}{3}(x^2-1)^{3/2} + C \qquad \text{Substitute for } u.$$

You can check this result by differentiating.

TRY IT 2

Find $\int x\sqrt{x^2+4}\,dx$.

EXAMPLE 3 Integration by Substitution

Find $\int \frac{e^{3x}}{1+e^{3x}}\,dx$.

SOLUTION Consider the substitution $u = 1 + e^{3x}$, which produces $du = 3e^{3x}\,dx$. To create $3e^{3x}\,dx$ as part of the integral, multiply and divide by 3.

$$\int \frac{e^{3x}}{1+e^{3x}}\,dx = \frac{1}{3}\int \underbrace{\frac{1}{1+e^{3x}}}_{1/u}\underbrace{3e^{3x}\,dx}_{du} \qquad \text{Multiply and divide by 3.}$$

$$= \frac{1}{3}\int \frac{1}{u}\,du \qquad \text{Substitute for } x \text{ and } dx.$$

$$= \frac{1}{3}\ln|u| + C \qquad \text{Log Rule}$$

$$= \frac{1}{3}\ln(1+e^{3x}) + C \qquad \text{Substitute for } u.$$

Note that the absolute value is not necessary in the final answer because the quantity $(1 + e^{3x})$ is positive for all values of x.

TRY IT 3

Find $\int \frac{e^{2x}}{1+e^{2x}}\,dx$.

EXAMPLE 4 Integration by Substitution

Find the indefinite integral.

$$\int x\sqrt{x-1}\,dx$$

SOLUTION Consider the substitution $u = x - 1$, which produces $du = dx$ and $x = u + 1$.

$$\begin{aligned}\int x\sqrt{x-1}\,dx &= \int (u+1)(u^{1/2})\,du && \text{Substitute for } x \text{ and } dx.\\ &= \int (u^{3/2} + u^{1/2})\,du && \text{Multiply.}\\ &= \frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2} + C && \text{Power Rule}\\ &= \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C && \text{Substitute for } u.\end{aligned}$$

This form of the antiderivative can be further simplified.

$$\begin{aligned}\frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C &= \frac{6}{15}(x-1)^{5/2} + \frac{10}{15}(x-1)^{3/2} + C\\ &= \frac{2}{15}(x-1)^{3/2}[3(x-1)+5] + C\\ &= \frac{2}{15}(x-1)^{3/2}(3x+2) + C\end{aligned}$$

You can check this answer by differentiating.

TRY IT 4

Find the indefinite integral.

$$\int x\sqrt{x+2}\,dx$$

Example 4 demonstrates one of the characteristics of integration by substitution. That is, you can often simplify the form of the antiderivative as it exists immediately after resubstitution into x-variable form. So, when working the exercises in this section, don't assume that your answer is incorrect just because it doesn't look exactly like the answer given in the back of the text. You may be able to reconcile the two answers by algebraic simplification.

TECHNOLOGY

If you have access to a symbolic integration utility, try using it to find an antiderivative of $f(x) = x^2\sqrt{x+1}$ and check your answer analytically using the substitution $u = x + 1$. You can also use the utility to solve several of the exercises in this section.

Substitution and Definite Integrals

The fourth step outlined in the guidelines for integration by substitution on page 389 suggests that you convert back to the variable x. To evaluate *definite* integrals, however, it is often more convenient to determine the limits of integration for the variable u. This is often easier than converting back to the variable x and evaluating the antiderivative at the original limits.

EXAMPLE 5 Using Substitution with a Definite Integral

Evaluate $\displaystyle\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$.

SOLUTION Use the substitution $u = \sqrt{2x-1}$, which implies that $u^2 = 2x - 1$, $x = \frac{1}{2}(u^2 + 1)$, and $dx = u\,du$. Before substituting, determine the new upper and lower limits of integration.

Lower limit: When $x = 1$, $u = \sqrt{2(1) - 1} = 1$.

Upper limit: When $x = 5$, $u = \sqrt{2(5) - 1} = 3$.

Now, substitute and integrate, as shown.

Integration limits for x

Integration limits for u

$$\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx = \int_1^3 \frac{1}{u}\left(\frac{u^2+1}{2}\right)u\,du \qquad \text{Substitute for } x, dx, \text{ and limits of integration.}$$

$$= \frac{1}{2}\int_1^3 (u^2+1)\,du \qquad \text{Simplify.}$$

$$= \frac{1}{2}\left[\frac{u^3}{3} + u\right]_1^3 \qquad \text{Find antiderivative.}$$

$$= \frac{1}{2}\left[(9+3) - \left(\frac{1}{3} + 1\right)\right] \qquad \text{Apply Fundamental Theorem.}$$

$$= \frac{16}{3} \qquad \text{Simplify.}$$

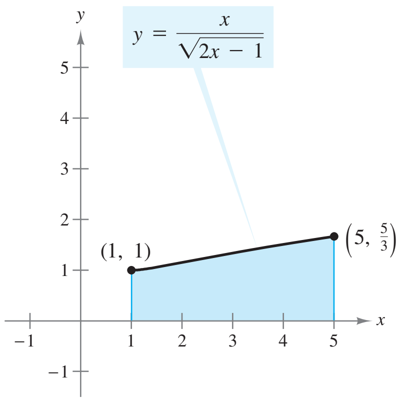


FIGURE 6.1 Region Before Substitution

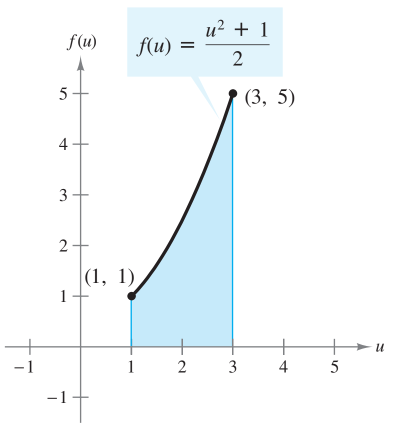


FIGURE 6.2 Region After Substitution

TRY IT 5

Evaluate $\displaystyle\int_0^2 x\sqrt{4x+1}\,dx$.

STUDY TIP

In Example 5, you can interpret the equation

$$\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx = \int_1^3 \frac{1}{u}\left(\frac{u^2+1}{2}\right)u\,du$$

graphically to mean that the two different regions shown in Figures 6.1 and 6.2 have the same area.

Application

Integration can be used to find the probability that an event will occur. In such an application, the real-life situation is modeled by a *probability density function f*, and the probability that x will lie between a and b is represented by

$$P(a \le x \le b) = \int_a^b f(x)\,dx.$$

The probability $P(a \le x \le b)$ must be a number between 0 and 1.

Researchers such as psychologists use definite integrals to represent the probability that an event will occur. For instance, a probability of 0.5 means that an event will occur about 50% of the time.

EXAMPLE 6 Finding a Probability

A psychologist finds that the probability that a participant in a memory experiment will recall between a and b percent (in decimal form) of the material is

$$P(a \le x \le b) = \int_a^b \frac{28}{9} x\sqrt[3]{1-x}\,dx, \quad 0 \le a \le b \le 1.$$

Find the probability that a randomly chosen participant will recall between 0% and 87.5% of the material.

SOLUTION Let $u = \sqrt[3]{1-x}$. Then $u^3 = 1 - x$, $x = 1 - u^3$, and $dx = -3u^2\,du$.

Lower limit: When $x = 0$, $u = \sqrt[3]{1-0} = 1$.

Upper limit: When $x = 0.875$, $u = \sqrt[3]{1-0.875} = 0.5$.

To find the probability, substitute and integrate, as shown.

$$\begin{aligned}\int_0^{0.875} \frac{28}{9} x\sqrt[3]{1-x}\,dx &= \int_1^{1/2} \left[\frac{28}{9}(1-u^3)(u)(-3u^2)\right] du \\ &= \left(\frac{28}{3}\right)\int_1^{1/2} (u^6 - u^3)\,du \\ &= \frac{28}{3}\left[\frac{u^7}{7} - \frac{u^4}{4}\right]_1^{1/2} \\ &\approx 0.865\end{aligned}$$

So, the probability is about 86.5%, as indicated in Figure 6.3.

ALGEBRA REVIEW

For help on the algebra in Example 6, see Example 1(a) in the *Chapter 6 Algebra Review* on page 446.

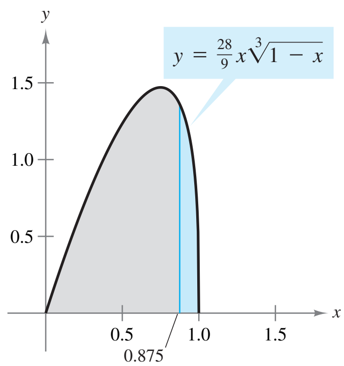


FIGURE 6.3

TRY IT 6

Use Example 6 to find the probability that a participant will recall between 0% and 62.5% of the material.

TAKE ANOTHER LOOK

Probability

In Example 6, explain how you could find a value of b such that $P(0 \le x \le b) = 0.5$.

PREREQUISITE REVIEW 6.1

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–8, evaluate the indefinite integral.

1. $\int 5\, dx$

2. $\int \frac{1}{3}\, dx$

3. $\int x^{3/2}\, dx$

4. $\int x^{2/3}\, dx$

5. $\int 2x(x^2 + 1)^3\, dx$

6. $\int 3x^2(x^3 - 1)^2\, dx$

7. $\int 6e^{6x}\, dx$

8. $\int \frac{2}{2x + 1}\, dx$

In Exercises 9–12, simplify the expression.

9. $2x(x - 1)^2 + x(x - 1)$

10. $6x(x + 4)^3 - 3x^2(x + 4)^2$

11. $3(x + 7)^{1/2} - 2x(x + 7)^{-1/2}$

12. $(x + 5)^{1/3} - 5(x + 5)^{-2/3}$

EXERCISES 6.1

In Exercises 1–38, find the indefinite integral.

1. $\int (x - 2)^4\, dx$

2. $\int (x + 5)^{3/2}\, dx$

3. $\int \frac{2}{(t - 9)^2}\, dt$

4. $\int \frac{4}{(1 - t)^3}\, dt$

5. $\int \frac{2t - 1}{t^2 - t + 2}\, dt$

6. $\int \frac{2y^3}{y^4 + 1}\, dy$

7. $\int \sqrt{1 + x}\, dx$

8. $\int (3 + x)^{5/2}\, dx$

9. $\int \frac{12x + 2}{3x^2 + x}\, dx$

10. $\int \frac{6x^2 + 2}{x^3 + x}\, dx$

11. $\int \frac{1}{(5x + 1)^3}\, dx$

12. $\int \frac{1}{(3x + 1)^2}\, dx$

13. $\int \frac{1}{\sqrt{x + 1}}\, dx$

14. $\int \frac{1}{\sqrt{5x + 1}}\, dx$

15. $\int \frac{e^{3x}}{1 - e^{3x}}\, dx$

16. $\int \frac{4e^{2x}}{1 + e^{2x}}\, dx$

17. $\int \frac{2x}{e^{3x^2}}\, dx$

18. $\int \frac{e^{\sqrt{x+1}}}{\sqrt{x + 1}}\, dx$

19. $\int \frac{x^2}{x - 1}\, dx$

20. $\int \frac{2x}{x - 4}\, dx$

21. $\int x\sqrt{x^2 + 4}\, dx$

22. $\int \frac{t}{\sqrt{1 - t^2}}\, dt$

23. $\int e^{5x}\, dx$

24. $\int te^{t^2+1}\, dt$

25. $\int \frac{e^{-x}}{e^{-x} + 2}\, dx$

26. $\int \frac{e^x}{1 + e^x}\, dx$

27. $\int \frac{x}{(x + 1)^4}\, dx$

28. $\int \frac{x^2}{(x + 1)^3}\, dx$

29. $\int \frac{x}{(3x - 1)^2}\, dx$

30. $\int \frac{5x}{(x - 4)^3}\, dx$

31. $\int \frac{1}{\sqrt{t} - 1}\, dt$

32. $\int \frac{1}{\sqrt{x} + 1}\, dx$

33. $\int \frac{2\sqrt{t} + 1}{t}\, dt$

34. $\int \frac{6x + \sqrt{x}}{x}\, dx$

35. $\int \frac{x}{\sqrt{2x + 1}}\, dx$

36. $\int \frac{x^2}{\sqrt{x - 1}}\, dx$

37. $\int t^2\sqrt{1 - t}\, dt$

38. $\int y^2\sqrt[3]{y + 1}\, dy$

In Exercises 39–46, evaluate the definite integral.

39. $\int_0^4 \sqrt{2x + 1}\, dx$

40. $\int_2^4 \sqrt{4x + 1}\, dx$

41. $\int_0^1 3xe^{x^2}\, dx$

42. $\int_0^2 e^{-2x}\, dx$

43. $\int_0^4 \frac{x}{(x + 4)^2}\, dx$

44. $\int_0^1 x(x + 5)^4\, dx$

45. $\int_0^{0.5} x(1 - x)^3\, dx$

46. $\int_0^{0.5} x^2(1 - x)^3\, dx$

In Exercises 47–54, find the area of the region bounded by the graphs of the equations. Then use a graphing utility to graph the region and verify your answer.

47. $y = x\sqrt{x-3},\ y = 0,\ x = 7$

48. $y = x\sqrt{2x+1},\ y = 0,\ x = 4$

49. $y = x^2\sqrt{1-x},\ y = 0,\ x = -3$

50. $y = x^2\sqrt{x+2},\ y = 0,\ x = 7$

51. $y = \dfrac{x^2-1}{\sqrt{2x-1}},\ y = 0,\ x = 1,\ x = 5$

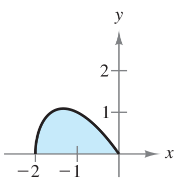
52. $y = \dfrac{2x-1}{\sqrt{x+3}},\ y = 0,\ x = \dfrac{1}{2},\ x = 6$

53. $y = x\sqrt[3]{x+1},\ y = 0,\ x = 0,\ x = 7$

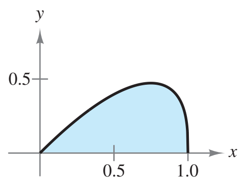
54. $y = x\sqrt[3]{x-2},\ y = 0,\ x = 2,\ x = 10$

In Exercises 55–58, find the area of the region bounded by the graphs of the equations.

55. $y = -x\sqrt{x+2},\ y = 0$

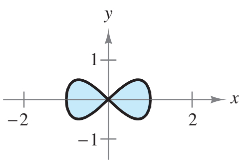


56. $y = x\sqrt[3]{1-x},\ y = 0$

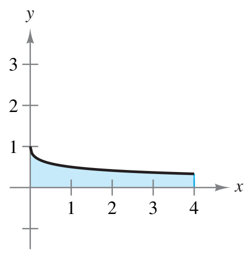


57. $y^2 = x^2(1-x^2)$

(*Hint:* Find the area of the region bounded by $y = x\sqrt{1-x^2}$ and $y = 0$. Then multiply by 4.)



58. $y = 1/(1+\sqrt{x})$, $y = 0,\ x = 0,\ x = 4$



In Exercises 59 and 60, find the volume of the solid generated by revolving the region bounded by the graph(s) of the equation(s) about the x-axis.

59. $y = x\sqrt{1-x^2}$

60. $y = \sqrt{x}(1-x)^2,\ y = 0$

In Exercises 61 and 62, find the average amount by which the function f exceeds the function g on the interval.

61. $f(x) = \dfrac{1}{x+1},\ g(x) = \dfrac{x}{(x+1)^2},\ [0, 1]$

62. $f(x) = x\sqrt{4x+1},\ g(x) = 2\sqrt{x^3},\ [0, 2]$

63. ***Probability*** The probability of recall in an experiment is modeled by

$$P(a \le x \le b) = \int_a^b \frac{15}{4}x\sqrt{1-x}\,dx$$

where x is the percent of recall (see figure).

(a) What is the probability of recalling between 40% and 80%?

(b) What is the median percent recall? That is, for what value of b is $P(0 \le x \le b) = 0.5$?

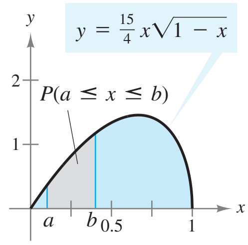


Figure for 63

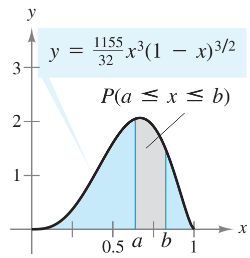


Figure for 64

64. ***Probability*** The probability of finding between a and b percent iron in ore samples is modeled by

$$P(a \le x \le b) = \int_a^b \frac{1155}{32}x^3(1-x)^{3/2}\,dx$$

(see figure). Find the probabilities that a sample will contain between (a) 0% and 25% and (b) 50% and 100% iron.

65. ***Meteorology*** During a two-week period in March in a small town near Lake Erie, the measurable snowfall S (in inches) on the ground can be modeled by

$$S(t) = t\sqrt{14-t}, \quad 0 \le t \le 14$$

where t represents the day.

(a) Use a graphing utility to graph the function.

(b) Find the average amount of snow on the ground during the two-week period.

(c) Find the total snowfall over the two-week period.

66. ***Revenue*** A company sells a seasonal product that generates a daily revenue R (in dollars per day) modeled by

$$R = 0.06t^2(365-t)^{1/2} + 1250, \quad 0 \le t \le 365$$

where t represents the day.

(a) Find the average daily revenue over a period of 1 year.

(b) Describe a product whose seasonal sales pattern resembles the model. Explain your reasoning.

In Exercises 67 and 68, use a program similar to the Midpoint Rule program on page 366 with $n = 10$ to approximate the area of the region bounded by the graph(s) of the equation(s).

67. $y = \sqrt[3]{x}\sqrt{4-x}, \quad y = 0$

68. $y^2 = x^2(1-x^2)$