8.6 L'HÔPITAL'S RULE

- Approximate limits that produce indeterminate forms.
- Use L'Hôpital's Rule to evaluate limits.

Indeterminate Forms and Limits

In Sections 1.5 and 3.6, you studied limits such as

$$\lim_{x\to 1} \frac{x^2 - 1}{x - 1}$$

and

$$\lim_{x\to\infty} \frac{2x + 1}{x + 1}.$$

In those sections, you discovered that direct substitution can produce an **indeterminate form** such as $0/0$ or ∞/∞. For instance, if you substitute $x = 1$ into the first limit, you obtain

$$\lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \frac{0}{0} \qquad \text{Indeterminate form}$$

which tells you nothing about the limit. To find the limit, you can factor and divide out like factors, as shown.

$$\begin{aligned}
\lim_{x\to 1} \frac{x^2 - 1}{x - 1} &= \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} && \text{Factor.}\\
&= \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} && \text{Divide out like factors.}\\
&= \lim_{x\to 1} (x+1) && \text{Simplify.}\\
&= 1 + 1 && \text{Direct substitution}\\
&= 2 && \text{Simplify.}
\end{aligned}$$

For the second limit, direct substitution produces the indeterminate form ∞/∞, which again tells you nothing about the limit. To evaluate this limit, you can divide the numerator and denominator by x. Then you can use the fact that the limit of $1/x$, as $x \to \infty$, is 0.

$$\begin{aligned}
\lim_{x\to\infty} \frac{2x + 1}{x + 1} &= \lim_{x\to\infty} \frac{2 + (1/x)}{1 + (1/x)} && \text{Divide numerator and denominator by } x.\\
&= \frac{2 + 0}{1 + 0} && \text{Evaluate limits.}\\
&= 2 && \text{Simplify.}
\end{aligned}$$

Algebraic techniques such as these tend to work well as long as the function itself is algebraic. To find the limits of other types of functions, such as exponential functions or trigonometric functions, you generally need to use a different approach.

EXAMPLE 1 Approximating a Limit

Find the limit.

$$\lim_{x\to 0} \frac{e^{3x} - 1}{x}$$

SOLUTION When evaluating a limit, you can choose from three basic approaches. That is, you can attempt to find the limit *analytically*, *graphically*, or *numerically*. For this particular limit, it is not clear how to use an analytic approach because direct substitution yields an indeterminate form.

$$\lim_{x\to 0} \frac{e^{3x} - 1}{x} = \lim_{x\to 0} \frac{e^{3(0)} - 1}{0} \qquad \text{Indeterminate form}$$

$$= \frac{0}{0} \qquad \text{Indeterminate form}$$

Using a graphical approach, you can graph the function, as shown in Figure 8.36, and then use the *zoom* and *trace* features to estimate the limit. Using a numerical approach, you can construct a table, such as that shown below.

x	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01
$\frac{e^{3x} - 1}{x}$	2.9554	2.9955	2.9996	?	3.0005	3.0045	3.0455

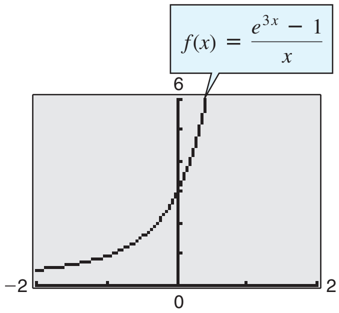

FIGURE 8.36

From the values in the table, it appears that the limit is 3. So, from either a graphical or a numerical approach, you can approximate the limit to be

$$\lim_{x\to 0} \frac{e^{3x} - 1}{x} = 3.$$

TRY IT 1

Find the limit.

$$\lim_{x\to 0} \frac{1 - e^{2x}}{x}$$

EXAMPLE 2 Approximating a Limit

Find the limit.

$$\lim_{x\to 0} \frac{\sin 4x}{x}$$

SOLUTION As in Example 1, it is not clear how to use an analytic approach because direct substitution yields an indeterminate form.

$$\lim_{x\to 0} \frac{\sin 4x}{x} = \lim_{x\to 0} \frac{\sin 4(0)}{0} = \frac{0}{0} \qquad \text{Indeterminate form}$$

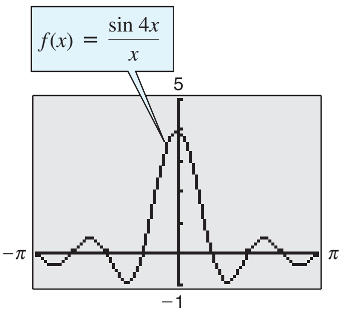

FIGURE 8.37

Using a graphical approach, you can graph the function, as shown in Figure 8.37, and then, using the *zoom* and *trace* features, you can estimate the limit to be 4. A numerical approach would lead to the same conclusion. So, using either a graphical or a numerical approach, you can approximate the limit to be

$$\lim_{x\to 0} \frac{\sin 4x}{x} = 4.$$

TRY IT 2

Find the limit.

$$\lim_{x\to 0} \frac{\sin(x/2)}{x}$$

L'Hôpital's Rule

L'Hôpital's Rule, which is named after the French mathematician Guillaume Francois Antoine de L'Hôpital (1661–1704), describes an analytic approach for evaluating limits.

L'Hôpital's Rule

Let (a, b) be an interval that contains c. Let f and g be differentiable in (a, b), except possibly at c. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$ or ∞/∞, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists or is infinite. The indeterminate form ∞/∞ comes in four forms: ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, and $(-\infty)/(-\infty)$. L'Hôpital's Rule can be applied to each of these forms.

DISCOVERY

Use a numerical or a graphical approach to approximate each limit.

(a) $\displaystyle\lim_{x \to 0} \frac{2^{2x} - 1}{x}$

(b) $\displaystyle\lim_{x \to 0} \frac{3^{2x} - 1}{x}$

(c) $\displaystyle\lim_{x \to 0} \frac{4^{2x} - 1}{x}$

(d) $\displaystyle\lim_{x \to 0} \frac{5^{2x} - 1}{x}$

What pattern do you observe? Does an analytic approach have an advantage for these limits? If so, explain your reasoning.

EXAMPLE 3 Using L'Hôpital's Rule

Find the limit.

$$\lim_{x \to 0} \frac{e^{3x} - 1}{x}$$

SOLUTION In Example 1, it was shown that the limit appears to be 3. Because direct substitution produces the indeterminate form $0/0$, you can apply L'Hôpital's Rule to obtain the same result.

$$\lim_{x \to 0} \frac{e^{3x} - 1}{x} = \lim_{x \to 0} \frac{\frac{d}{dx}[e^{3x} - 1]}{\frac{d}{dx}[x]}$$ Apply L'Hôpital's Rule.

$$= \lim_{x \to 0} \frac{3e^{3x}}{1}$$ Differentiate numerator and denominator separately.

$$= \frac{3e^{3(0)}}{1}$$ Direct substitution

$$= 3$$ Simplify.

TRY IT 3

Find the limit using L'Hôpital's Rule.

$$\lim_{x \to 0} \frac{1 - e^{2x}}{x}$$

STUDY TIP

Be sure you see that L'Hôpital's Rule involves $f'(x)/g'(x)$, *not* the derivative of the quotient $f(x)/g(x)$.

STUDY TIP

You *cannot* apply L'Hôpital's Rule unless direct substitution produces the indeterminate form $0/0$ or ∞/∞. For instance, if you were to apply L'Hôpital's Rule to the limit

$$\lim_{x\to 0} \frac{x}{e^x}$$

you would incorrectly conclude that the limit is 1. Can you see why this is incorrect?

TRY IT 4

Find the limit using L'Hôpital's Rule.

$$\lim_{x\to 0} \frac{\sin(x/2)}{x}$$

EXAMPLE 4 Using L'Hôpital's Rule

Find the limit.

$$\lim_{x\to 0} \frac{\sin 4x}{x}$$

SOLUTION In Example 2, it was shown that the limit appears to be 4. Because direct substitution produces the indeterminate form $0/0$

$$\lim_{x\to 0} \frac{\sin 4x}{x} = \frac{0}{0} \qquad \text{Indeterminate form}$$

you can apply L'Hôpital's Rule to obtain the same result.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin 4x}{x} &= \lim_{x\to 0} \frac{\frac{d}{dx}[\sin 4x]}{\frac{d}{dx}[x]} && \text{Apply L'Hôpital's Rule.} \\
&= \lim_{x\to 0} \frac{4\cos 4x}{1} && \text{Differentiate numerator and denominator separately.} \\
&= \frac{4\cos[4(0)]}{1} && \text{Direct substitution} \\
&= 4 && \text{Simplify.}
\end{aligned}$$

STUDY TIP

Another form of L'Hôpital's Rule states that if the limit of $f(x)/g(x)$ as x approaches ∞ or $-\infty$ produces the indeterminate form $0/0$ or ∞/∞, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

This form is used in Example 5.

TRY IT 5

Find the limit.

$$\lim_{x\to\infty} \frac{e^{2x}}{e^{4x}+1}$$

EXAMPLE 5 Using L'Hôpital's Rule

Find the limit.

$$\lim_{x\to\infty} \frac{e^x}{e^{2x}+1}$$

SOLUTION Because direct substitution produces the indeterminate form ∞/∞

$$\lim_{x\to\infty} \frac{e^x}{e^{2x}+1} = \frac{\infty}{\infty} \qquad \text{Indeterminate form}$$

you can apply L'Hôpital's Rule, as shown.

$$\begin{aligned}
\lim_{x\to\infty} \frac{e^x}{e^{2x}+1} &= \lim_{x\to\infty} \frac{\frac{d}{dx}[e^x]}{\frac{d}{dx}[e^{2x}+1]} && \text{Apply L'Hôpital's Rule.} \\
&= \lim_{x\to\infty} \frac{e^x}{2e^{2x}} && \text{Differentiate numerator and denominator separately.} \\
&= \lim_{x\to\infty} \frac{1}{2e^x} && \text{Simplify.} \\
&= 0 && \text{Evaluate limit.}
\end{aligned}$$

Sometimes it is necessary to apply L'Hôpital's Rule more than once to remove an indeterminate form. This is shown in Example 6.

EXAMPLE 6 Using L'Hôpital's Rule Repeatedly

Find the limit.

$$\lim_{x\to-\infty} \frac{x^2}{e^{-x}}$$

SOLUTION Because direct substitution results in the indeterminate form ∞/∞

$$\lim_{x\to-\infty} \frac{x^2}{e^{-x}} = \frac{\infty}{\infty} \qquad \text{Indeterminate form}$$

you can apply L'Hôpital's Rule, as shown.

$$\lim_{x\to-\infty} \frac{x^2}{e^{-x}} = \lim_{x\to-\infty} \frac{\frac{d}{dx}[x^2]}{\frac{d}{dx}[e^{-x}]} \qquad \text{Apply L'Hôpital's Rule.}$$

$$= \lim_{x\to-\infty} \frac{2x}{-e^{-x}} \qquad \text{Differentiate numerator and denominator separately.}$$

$$= \frac{-\infty}{-\infty} \qquad \text{Indeterminate form}$$

After one application of L'Hôpital's Rule, you still obtain an indeterminate form. In such cases, you can try L'Hôpital's Rule again, as shown.

$$\lim_{x\to-\infty} \frac{2x}{-e^{-x}} = \lim_{x\to-\infty} \frac{\frac{d}{dx}[2x]}{\frac{d}{dx}[-e^{-x}]} \qquad \text{Apply L'Hôpital's Rule.}$$

$$= \lim_{x\to-\infty} \frac{2}{e^{-x}} \qquad \text{Differentiate numerator and denominator separately.}$$

$$= 0 \qquad \text{Evaluate limit.}$$

So, you can conclude that the limit is zero.

TRY IT 6

Find the limit.

$$\lim_{x\to\infty} \frac{x^2}{e^{2x}}$$

STUDY TIP

Remember that even in those cases in which L'Hôpital's Rule can be applied to determine a limit, it is still a good idea to confirm the result graphically or numerically. For instance, the table below provides a numerical confirmation of the limit in Example 6.

x	-2	-4	-6	-8	-10	-12
$\frac{x^2}{e^{-x}}$	0.5413	0.2931	0.0892	0.0215	0.0045	0.0008

L'Hôpital's Rule can be used to compare the rates of growth of two functions. For instance, consider the limit in Example 5

$$\lim_{x\to\infty} \frac{e^x}{e^{2x}+1} = 0.$$

Both of the functions $f(x) = e^x$ and $g(x) = e^{2x} + 1$ approach infinity as $x \to \infty$. However, because the quotient $f(x)/g(x)$ approaches 0 as $x \to \infty$, it follows that the denominator is growing much more rapidly than the numerator.

STUDY TIP

L'Hôpital's Rule is necessary to solve certain real-life problems such as compound interest problems and other business applications.

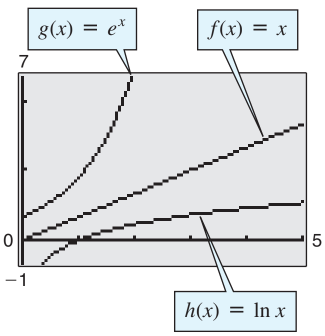

FIGURE 8.38

EXAMPLE 7 Comparing Rates of Growth

Each of the functions below approaches infinity as x approaches infinity. Which function has the highest rate of growth?

(a) $f(x) = x$ (b) $g(x) = e^x$ (c) $h(x) = \ln x$

SOLUTION Using L'Hôpital's Rule, you can show that each of the limits is zero.

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{\frac{d}{dx}[x]}{\frac{d}{dx}[e^x]} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}[\ln x]}{\frac{d}{dx}[x]} = \lim_{x\to\infty} \frac{1}{x} = 0$$

$$\lim_{x\to\infty} \frac{\ln x}{e^x} = \lim_{x\to\infty} \frac{\frac{d}{dx}[\ln x]}{\frac{d}{dx}[e^x]} = \lim_{x\to\infty} \frac{1}{xe^x} = 0$$

From this, you can conclude that $h(x) = \ln x$ has the lowest rate of growth, $f(x) = x$ has a higher rate of growth, and $g(x) = e^x$ has the highest rate of growth. This conclusion is confirmed graphically in Figure 8.38.

TRY IT 7

Use a graphing utility to order the functions according to the rate of growth of each function as x approaches infinity.

(a) $f(x) = e^{2x}$

(b) $g(x) = x^2$

(c) $h(x) = \ln x^2$

TAKE ANOTHER LOOK

Comparing Rates of Growth

Each of the functions below approaches infinity as x approaches infinity. In each pair, which function has the higher rate of growth? Use a graphing utility to verify your choice.

a. $f(x) = e^x$, $g(x) = e^{\sqrt{x}}$ b. $f(x) = \sqrt{x^3+1}$, $g(x) = x$ c. $f(x) = \tan \dfrac{\pi x}{2x+1}$, $g(x) = x$

PREREQUISITE REVIEW 8.6

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–8, find the limit, if it exists.

1. $\lim\limits_{x \to \infty} (x^2 - 10x + 1)$

2. $\lim\limits_{x \to \infty} \dfrac{2x + 5}{3}$

3. $\lim\limits_{x \to \infty} \dfrac{x + 1}{x^2}$

4. $\lim\limits_{x \to \infty} \dfrac{5x - 2}{x^2 + 1}$

5. $\lim\limits_{x \to \infty} \dfrac{2x^2 - 5x + 7}{3x^2 + 12x + 4}$

6. $\lim\limits_{x \to \infty} \dfrac{x^2 - 1}{x^2 + 4}$

7. $\lim\limits_{x \to \infty} \dfrac{x^2 - 5}{2x + 7}$

8. $\lim\limits_{x \to \infty} \dfrac{x^2 - 2x + 1}{x - 4}$

In Exercises 9–12, find the first derivative of the function.

9. $f(x) = \cos x^2$

10. $f(x) = \sin(5x - 1)$

11. $f(x) = \sec 4x$

12. $f(x) = \tan(x^2 - 2)$

In Exercises 13–16, find the second derivative of the function.

13. $f(x) = \sin(2x + 3)$

14. $f(x) = \cos \dfrac{x}{2}$

15. $f(x) = \tan x$

16. $f(x) = \cot x$

EXERCISES 8.6

In Exercises 1–6, decide whether the limit produces an indeterminate form.

1. $\lim\limits_{x \to 0} \dfrac{2x + \sqrt{x}}{x}$

2. $\lim\limits_{x \to \infty} \dfrac{x^2 + 4x - 3}{7x^2 + 2}$

3. $\lim\limits_{x \to -\infty} \dfrac{4}{x^2 + e^x}$

4. $\lim\limits_{x \to 0} \dfrac{\sin x}{e^x}$

5. $\lim\limits_{x \to \infty} \dfrac{2xe^{2x}}{3e^x}$

6. $\lim\limits_{x \to \infty} \dfrac{\ln x}{x}$

In Exercises 7–10, complete the table to estimate the limit numerically.

7. $\lim\limits_{x \to 0} \dfrac{e^{-x} - 1}{3x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$				?			

8. $\lim\limits_{x \to 3} \dfrac{x^2 + x - 12}{x^2 - 9}$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$				?			

9. $\lim\limits_{x \to 0} \dfrac{\sin x}{5x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$				?			

10. $\lim\limits_{x \to 0} \dfrac{\cos 2x - 1}{6x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$				?			

In Exercises 11–16, use a graphing utility to find the indicated limit graphically.

11. $\lim\limits_{x \to \infty} \dfrac{\ln x}{x^2}$

12. $\lim\limits_{x \to \infty} \dfrac{e^x}{x^2}$

13. $\lim\limits_{x \to 1} \dfrac{\ln(2 - x)}{x - 1}$

14. $\lim\limits_{x \to 1} \dfrac{e^{x-1}}{x^2 - 1}$

15. $\lim\limits_{x \to 2} \dfrac{x^2 - x - 2}{x^2 - 5x + 6}$

16. $\lim\limits_{x \to 1} \dfrac{x^2 + 3x - 4}{x^2 + 2x - 3}$

In Exercises 17–48, use L'Hôpital's Rule to find the limit. You may need to use L'Hôpital's Rule repeatedly.

17. $\lim\limits_{x\to 0} \dfrac{e^{-x}-1}{x}$

18. $\lim\limits_{x\to 3} \dfrac{x^2+x-12}{x^2-9}$

19. $\lim\limits_{x\to 0} \dfrac{\sin x}{5x}$

20. $\lim\limits_{x\to 0} \dfrac{\cos 2x - 1}{6x}$

21. $\lim\limits_{x\to \infty} \dfrac{\ln x}{x^2}$

22. $\lim\limits_{x\to \infty} \dfrac{e^x}{x^2}$

23. $\lim\limits_{x\to 2} \dfrac{x^2-x-2}{x^2-5x+6}$

24. $\lim\limits_{x\to 1} \dfrac{x^2+3x-4}{x^2+2x-3}$

25. $\lim\limits_{x\to 0} \dfrac{2x+1-e^x}{x}$

26. $\lim\limits_{x\to 0} \dfrac{2x-1+e^{-x}}{3x}$

27. $\lim\limits_{x\to \infty} \dfrac{\ln x}{e^x}$

28. $\lim\limits_{x\to \infty} \dfrac{3x}{e^x}$

29. $\lim\limits_{x\to \infty} \dfrac{4x^2+2x-1}{3x^2-7}$

30. $\lim\limits_{x\to \infty} \dfrac{3x^2+5}{2x^2-11}$

31. $\lim\limits_{x\to \infty} \dfrac{1-x}{e^x}$

32. $\lim\limits_{x\to 0} \dfrac{e^{3x}-1}{2x}$

33. $\lim\limits_{x\to 1} \dfrac{\ln x}{x^2-1}$

34. $\lim\limits_{x\to 2} \dfrac{\ln(x-1)}{x-2}$

35. $\lim\limits_{x\to 0} \dfrac{\sin 2x}{\sin 5x}$

36. $\lim\limits_{x\to 0} \dfrac{\tan 2x}{x}$

37. $\lim\limits_{x\to \pi/2} \dfrac{2\cos x}{3\cos x}$

38. $\lim\limits_{x\to 0} \dfrac{\cos 2x - 1}{x}$

39. $\lim\limits_{x\to 0} \dfrac{\sin x}{e^x-1}$

40. $\lim\limits_{x\to 1} \dfrac{\ln x}{1-x^2}$

41. $\lim\limits_{x\to \pi} \dfrac{\cos(x/2)}{x-\pi}$

42. $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{x}-1}$

43. $\lim\limits_{x\to \infty} \dfrac{x}{\sqrt{x+1}}$

44. $\lim\limits_{x\to 0} \dfrac{\sqrt{x+4}-2}{x}$

45. $\lim\limits_{x\to 0} \dfrac{\sqrt{4-x^2}-2}{x}$

46. $\lim\limits_{x\to \infty} \dfrac{x^2+2}{e^{x^2}}$

47. $\lim\limits_{x\to \infty} \dfrac{e^{3x}}{x^3}$

48. $\lim\limits_{x\to \infty} \dfrac{x^2}{e^x \ln x}$

In Exercises 49–56, find the limit. (*Hint:* L'Hôpital's Rule does not apply in every case.)

49. $\lim\limits_{x\to \infty} \dfrac{x^2+2x+1}{x^2+3}$

50. $\lim\limits_{x\to \infty} \dfrac{3x^3-7x+5}{4x^3+2x-11}$

51. $\lim\limits_{x\to -1} \dfrac{x^3+3x^2-6x-8}{2x^3-3x^2-5x+6}$

52. $\lim\limits_{x\to 0} \dfrac{2x^3-x^2-3x}{x^4+2x^3-9x^2-18x}$

53. $\lim\limits_{x\to 3} \dfrac{\ln(x-2)}{x-2}$

54. $\lim\limits_{x\to -1} \dfrac{\ln(x+2)}{x+2}$

55. $\lim\limits_{x\to 1} \dfrac{2\ln x}{e^x}$

56. $\lim\limits_{x\to 0} \dfrac{\sin \pi x}{x}$

In Exercises 57–62, use L'Hôpital's Rule to compare the rates of growth of the numerator and the denominator.

57. $\lim\limits_{x\to \infty} \dfrac{x^2}{e^{4x}}$

58. $\lim\limits_{x\to \infty} \dfrac{x^3}{e^{2x}}$

59. $\lim\limits_{x\to \infty} \dfrac{(\ln x)^4}{x}$

60. $\lim\limits_{x\to \infty} \dfrac{(\ln x)^2}{x^4}$

61. $\lim\limits_{x\to \infty} \dfrac{(\ln x)^n}{x^m}$

62. $\lim\limits_{x\to \infty} \dfrac{x^m}{e^{nx}}$

63. Complete the table to show that x eventually "overpowers" $(\ln x)^5$.

x	10	10^2	10^3	10^4	10^5	10^6
$\dfrac{(\ln x)^5}{x}$						

64. Complete the table to show that e^x eventually "overpowers" x^6.

x	1	2	4	8	12	20	30
$\dfrac{e^x}{x^6}$							

In Exercises 65–68, L'Hôpital's Rule is used incorrectly. Describe the error.

65. $\lim\limits_{x\to 0} \dfrac{e^{3x}-1}{e^x} = \lim\limits_{x\to 0} \dfrac{3e^{3x}}{e^x} = \lim\limits_{x\to 0} 3e^{2x} = 3$

66. $\lim\limits_{x\to 0} \dfrac{\sin \pi x + 1}{x} = \lim\limits_{x\to 0} \dfrac{\pi \cos \pi x}{1} = \pi$

67. $\lim\limits_{x\to 1} \dfrac{e^x-1}{\ln x} = \lim\limits_{x\to 1} \dfrac{e^x}{(1/x)} = \lim\limits_{x\to 1} xe^x = e$

68. $\lim\limits_{x\to \infty} \dfrac{e^{-x}}{1-e^{-x}} = \lim\limits_{x\to \infty} \dfrac{-e^{-x}}{e^{-x}} = \lim\limits_{x\to \infty} -1 = -1$

In Exercises 69–72, use a graphing utility to (a) graph the function and (b) find the limit (if it exists).

69. $\lim_{x\to 2} \dfrac{x-2}{\ln(3x-5)}$

70. $\lim_{x\to\infty} \dfrac{x^4}{e^x}$

71. $\lim_{x\to -2} \dfrac{\sqrt{x^2-4}-5}{x+2}$

72. $\lim_{x\to 0} \dfrac{3\tan x}{5x}$

73. Show that L'Hôpital's Rule fails for the limit

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}.$$

What is the limit?

74. Use a graphing utility to graph

$$f(x) = \frac{x^k - 1}{k}$$

for $k = 1, 0.1$, and 0.01. Then find

$$\lim_{k\to 0^+} \frac{x^k - 1}{k}.$$

75. ***Sales*** The sales (in millions of dollars per year) for the years 1994 through 2003 for two major drugstore chains are modeled by

$$f(t) = -1673.4 + 449.78t^2 - 28.548t^3 + 0.0113e^t$$

Rite Aid

$$g(t) = 5947.9 - 3426.94t + 841.872t^2 - 35.4110t^3$$

CVS

where t is the year, with $t = 4$ corresponding to 1994. *(Source: Rite Aid Corporation; CVS Corporation)*

(a) Use a graphing utility to graph both models for $4 \le t \le 13$.

(b) Use your graph to determine which company had the larger rate of growth of sales for $4 \le t \le 13$.

(c) Use your knowledge of functions to predict which company ultimately will have the larger rate of growth. Explain your reasoning.

(d) If the models continue to be valid after 2004, when will one company's sales surpass the other company's sales and always be greater?

76. ***Compound Interest*** The formula for the amount A in a savings account compounded n times per year for t years at an interest rate r and an initial deposit of P is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Use L'Hôpital's Rule to show that the limiting formula as the number of compoundings per year becomes infinite is $A = Pe^{rt}$.

77. ***Research Project*** Use your school's library, the Internet, or some other reference source to gather data on the sales growth of two competing companies. Model the data for both companies, and determine which company is growing at a higher rate. Write a short paper that describes your findings, including the factors that account for the growth of the faster-growing company.

True or False? In Exercises 78–81, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

78. $\lim_{x\to 0} \dfrac{x^2+3x-1}{x+1} = \lim_{x\to 0} \dfrac{2x+3}{1} = 3$

79. $\lim_{x\to\infty} \dfrac{x}{1-x} = \lim_{x\to\infty} \dfrac{1}{-1} = 1$

80. If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0,$$

then $g(x)$ has a higher growth rate than $f(x)$.

81. If $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 1$, then $g(x) = f(x)$.

BUSINESS CAPSULE

Courtesy of Align Technology, Inc.

Two graduates of Stanford University, Zia Christi and Kelsey Wirth, developed an alternative to metal braces: a clear, removable aligner. They use 3-D graphics to mold a person's teeth and to prepare a series of aligners that gently shift the teeth to a desired final position. The company they created, Align Technology, had total revenues of \$6.7 million in 2000 and \$122.7 million in 2003, with over 44,000 patients using the aligners. The aligners provide orthodontists with an alternative treatment plan and provide patients with convenience.