1. Evaluate the following indefinite integrals.

(a) 
\[ \int \sin x \cos x \, dx \]
Let \[ u = \sin x \]
\[ du = \cos x \, dx \]
\[ = \int u \, du \]
\[ = \frac{1}{2} u^2 + C \]
\[ = \frac{1}{2} \sin^2 x + C \]

(b) 
\[ \int \frac{x - 1}{x + 1} \, dx \]
Let \[ u = x + 1 \]
\[ du = dx \]
\[ = \int \frac{u - 2}{u} \, du \]
\[ = \int 1 - \frac{2}{u} \, du \]
\[ = u - 2 \ln |u| + C \]
\[ = x + 1 - 2 \ln |x + 1| + C \]

2. Evaluate the following definite integrals.

(a) 
\[ \int_0^1 \sqrt{1 + 2r} \, dr \]
Let \[ u = 1 + 2r \]
\[ du = 2dr \]
\[ = \frac{1}{2} \int_1^3 \sqrt{u} \, du \]
\[ = \frac{1}{3} \left[ u^{3/2} \right]_1^3 \]
\[ = \frac{1}{3} (3^{3/2} - 1) \]

(b) 
\[ \int_{-1}^1 1 - \sqrt{1 - x^2} \, dx \]
From geometric considerations, the area under the graph is \( 2 - \pi/2 \).
3. \[ \int_{-1}^{4} |x| \, dx \]

From geometric considerations, the area under the graph is \( \frac{17}{2} \).

4. \[ \int e^x + e^{-x} + 2 \, dx \]

\[ u = e^x \]
\[ du = e^x dx = u \, dx \]

\[ = \int \frac{1}{u + 1/u + 2} \left( \frac{1}{u} \right) \, du \]
\[ = \int \frac{1}{u^2 + 2u + 1} \, du \]
\[ = \int \frac{1}{(u + 1)^2} \, du \]
\[ = -(u + 1)^{-1} + C \]
\[ = -(e^x + 1)^{-1} + C \]

5. For \( x \) with \( 0 \leq x \leq 1 \), define \( g(x) = \int_{0}^{\pi} t \sin^x t \, dt \). Find the maximum value of \( g \) over the interval \([0, 1]\).

**Solution.** Suppose that \( x \) is a number in \([0, 1]\). Since \( \sin t \leq 1 \) for all \( t \), we have
\[ (\sin t)^x \leq 1 = (\sin t)^0 \]
It follows that
\[ t (\sin t)^x \leq t (\sin t)^0 \]
for all \( t \) in \([0, \pi]\). Thus by the domination property (see property 7 in table 5.3 and also problem 5.3.71 from homework 2), we have
\[ \int_{0}^{\pi} t (\sin t)^x \, dt \leq \int_{0}^{\pi} t (\sin t)^0 \, dt, \]
i.e., \( g(x) \leq g(0) \). It follows that the max occurs at \( g(0) = \int_{0}^{\pi} t \, dt = \frac{1}{2} \pi^2 \).

6. Find \( \lim_{x \to 0} \frac{1}{2} \int_{0}^{\pi} \cos(t^2) \, dt \).

**Solution.** Let \( A(x) = \int_{0}^{\pi} \cos(t^2) \, dt \). Then by the Fundamental Theorem of Calculus, we have \( A'(x) = \cos(x^2) \). Hence
\[ \lim_{x \to 0} \frac{A(x)}{x} = \lim_{x \to 0} \frac{A(x) - A(0)}{x} \]
\[ = A'(0) \]
\[ = \cos(0^2) = 1 \]

Some students gave a solution using l’Hopital’s rule (see the solution to problem 5.4.76 from homework 2), which is similar to the solution given here.
7. The gas mileage of a car depends on its velocity. When the velocity is \( v \), the gas mileage is \( f(v) = e^{-v^2} \) miles/gallon. If the velocity at time \( t \) is \( v(t) = t \) miles/hour, how many gallons of gas are used after 5 hours. (Hint: Start by finding the amount of gas \( dg \) used between time \( t \) and \( t + dt \).)

**Solution.** Let

\[
g(t) = \text{gas used by time } t;
\]
\[
s(t) = \text{position at time } t.
\]

From the definition of velocity,

\[
ds = vdt. \tag{1}
\]

From the equation given for gas mileage,

\[
ds = e^{-v^2}dg \tag{2}
\]

Re-arranging terms in equations (1) and (2) gives

\[
dg = ve^{v^2}dt
\]
\[
= te^{t^2}dt,
\]

where the last line follows from the fact that \( v(t) = t \). The amount of gas used is

\[
\int dg = \int_0^5 te^{t^2} dt
\]
\[
= \frac{1}{2}e^{t^2}\big|_0^5
\]
\[
= \frac{1}{2}e^{25} - \frac{1}{2} \text{ gallons.}
\]

Although the calculations here are not difficult, this problem is more conceptual than the others and a majority of the class got 0 points. This was the only problem on the exam not inspired by a homework problem.