Mass Problems

1. Find the mass of the triangular region below. All lengths are in meters, and the density of the region is given by $\delta(x) = x$ grams/m$^2$. (Note that since the object is 2-dimensional, its density is its mass per unit area.)

\[\begin{array}{c}
\text{Solution. Since the density is a function of } x, \text{ we divide the region into thin vertical strips of thickness } dx \text{ as shown in the following figure:}
\end{array}\]

\[\begin{array}{c}
\text{(Question: What kind of strips would we use if the density were a function of } y?)
\end{array}\]

Since the height of the strip is $1 - x$, its area is $(1 - x)dx$, and hence its mass is

\[dm = (\text{density of strip}) \times (\text{area of strip}) = x(1 - x)dx\]

Hence the total mass of the triangle is

\[m = \int dm = \int_0^1 x(1 - x) \, dx = \frac{1}{2} x^2 - \frac{1}{3} x^3 \big|_0^1 = \frac{1}{6} \text{ grams}\]

2. Find the mass of the triangular region below. All lengths are in meters, and the density of the region is given by $\delta(x, y) = e^{(x+y)^2}$ grams/m$^2$. 
Solution. Note that we can write the density as \( \delta(r) = e^{r^2} \), where \( r = x + y \). For \( r \) in \([0, 1]\), the graph of \( x + y = r \) is a diagonal line that passes through the point \((r, 0)\). This suggests dividing the triangle into thin diagonal strips as shown in the following figure:

In the picture above, the strip is bounded by coordinate axes and the lines \( x + y = r \) and \( x + y = r + dr \). Its area is \( r\,dr \), which can be seen from the following calculation:

Let \( A(r) = \frac{1}{2}r^2 \) be the area of a triangle whose base and height are both \( r \), and let \( dA = A(r + dr) - A(r) \). Then

\[
\text{area of strip} = dA = r\,dr,
\]

where the last line follows from differentiation. It follows that the mass of the strip is

\[
dm = (\text{density of strip}) \times (\text{area of strip}) = e^{r^2}r\,dr
\]

Hence the total mass of the triangle is

\[
m = \int dm = \int_0^1 re^{r^2}dr = \frac{1}{2}e^{r^2}\bigg|_0^1 = \frac{1}{2}(e - 1) \text{ grams}
\]
3. A thin plate occupies the region of the plane bounded by the circle $x^2 + y^2 = 1$. Find the total mass if the density at the point $(x, y)$ is given by $\delta(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$. (Hint: divide the region into thin circular “rings” centered at the origin.)

Answer: $2\pi$.

4. The region bounded by the graph of $y = x^2$ and the $x$-axis, between 0 and 1, is revolved about the $x$-axis. The resulting solid has density given by $\delta(x) = x$. (Here the object is 3-dimensional, so its density is its mass per unit volume.) Find the total mass.

Answer: $\pi/6$. 