Math 21B-B - Homework Set 2

Section 5.3:

1. (a) \( \lim_{\|P\| \to 0} \sum_{k=1}^{n} (c_k^2 - 3c_k) \Delta x_k \), where \( P \) is a partition of \([-7, 5]\).

\[ \int_{-7}^{5} (x^2 - 3x) \, dx \]

(b) \( \lim_{\|P\| \to 0} \sum_{k=1}^{n} \sqrt{4-c_k^2} \Delta x_k \), where \( P \) is a partition of \([0, 1]\).

\[ \int_{0}^{1} \sqrt{4-x^2} \, dx \]

(c) \( \lim_{\|P\| \to 0} \sum_{k=1}^{n} (\tan c_k) \Delta x_k \), where \( P \) is a partition of \([0, \pi/4]\).

\[ \int_{0}^{\pi/4} \tan(x) \, dx \]

2. Suppose that \( f \) and \( g \) are integrable and that:

\[ \int_{1}^{2} f(x) \, dx = -4, \quad \int_{1}^{5} f(x) \, dx = 6, \quad \int_{1}^{5} g(x) = 8. \]

(a) \( \int_{2}^{1} g(x) \, dx = 0 \)

Zero Width - Rule 2

(b) \( \int_{1}^{5} g(x) \, dx = -\int_{1}^{5} g(x) \, dx = -8 \)

Order of Integration - Rule 1

(c) \( \int_{1}^{2} 3f(x) \, dx = 3 \int_{1}^{2} f(x) \, dx = 3 \cdot -4 = -12. \)

Constant Multiple - Rule 3

(d) \( \int_{2}^{5} f(x) \, dx = \int_{1}^{5} f(x) \, dx - \int_{1}^{2} f(x) \, dx = 6 - (-4) = 10 \)

Additivity - Rule 5

(e) \( \int_{1}^{5} [f(x) - g(x)] \, dx = \int_{1}^{5} f(x) \, dx - \int_{1}^{5} g(x) \, dx = 6 - 8 = -2 \)

Sum and Difference - Rule 4
\[
\int_1^5 [4f(x) - g(x)] \, dx = \int_1^5 4f(x) \, dx - \int_1^5 g(x) \, dx = 4\int_1^5 f(x) \, dx - \int_1^5 g(x) \, dx = 4 \cdot 6 - 8 = 24 - 8 = 16
\]

**Constant Multiple and Sum/Difference - Rules 3 and 4**

3. Suppose that \( f \) and \( h \) are integrable and that:

\[
\int_1^9 f(x) \, dx = -1, \quad \int_7^9 f(x) \, dx = 5, \quad \int_7^9 h(x) \, dx = 4.
\]

(a) \( \int_1^9 -2f(x) \, dx = -2\int_1^9 f(x) \, dx = -2 \cdot -1 = 2 \)

**Constant Multiple - Rule 3**

(b) \( \int_7^9 [f(x) + h(x)] \, dx = \int_7^9 f(x) \, dx + \int_7^9 h(x) \, dx = 5 + 4 = 9 \)

**Sum and Difference - Rule 4**

(c) \( \int_7^9 [2f(x) - 3h(x)] \, dx = \int_7^9 2f(x) \, dx - \int_7^9 3h(x) \, dx = 2\int_7^9 f(x) \, dx - 3\int_7^9 h(x) \, dx = 2 \cdot 5 - 3 \cdot 4 = 10 - 12 = -2 \)

**Constant Multiple and Sum/Difference - Rules 3 and 4**

(d) \( \int_1^1 f(x) \, dx = \int_1^9 f(x) \, dx = -(-1) = 1 \)

**Order of Integration - Rule 1**

(e) \( \int_1^7 f(x) \, dx = \int_1^9 f(x) \, dx - \int_7^9 f(x) \, dx = -1 - 5 = -6 \)

**Additivity - Rule 5**

(f) \( \int_7^9 [h(x) - f(x)] \, dx = \int_7^9 [h(x) - f(x)] \, dx = - \left[ \int_7^9 h(x) \, dx - \int_7^9 f(x) \, dx \right] = -4 - 5 = -(-1) = 1 \)

**Order of Integration and Sum/Difference - Rules 1 and 4**

4. \( \int_{-2}^1 |x| \, dx \)
\[ \int_{-2}^{1} |x| \, dx = \text{Area of left triangle} + \text{Area of right triangle} \]
\[ = \frac{1}{2} \cdot 2 \cdot 2 + \frac{1}{2} \cdot 1 \cdot 1 = 2 + 1/2 \]
\[ = 2 + 1/2 \]
\[ = 5/2 \text{ square units} \]

5. \[ \int_{-1}^{1} (1 + \sqrt{1 - x^2}) \, dx \]
\[ \int_{-1}^{1} (1 + \sqrt{1 - x^2}) \, dx = \text{Area of semicircle} + \text{Area of Rectangle} \]
\[ = \frac{1}{2} \cdot \pi \cdot 1^2 + 1 \cdot 2 \]
\[ = \frac{\pi}{2} + 2 \text{ square units} \]

6. (a) \( f(x) = x^2 - 1 \) on \([0, \sqrt{3}]\)

\[ \text{av}(f) = \left( \frac{1}{\sqrt{3} - 0} \right) \int_{0}^{\sqrt{3}} (x^2 - 1) \, dx \]
\[ = \left( \frac{1}{\sqrt{3}} \right) \left[ \int_{0}^{\sqrt{3}} x^2 \, dx - \int_{0}^{\sqrt{3}} 1 \, dx \right] \]
\[ = \left( \frac{1}{\sqrt{3}} \right) \left[ \frac{1}{3} x^3 \bigg|_{0}^{\sqrt{3}} - x \bigg|_{0}^{\sqrt{3}} \right] \]
\[ = \left( \frac{1}{\sqrt{3}} \right) \left[ \frac{1}{3} (3\sqrt{3} - 0) - (\sqrt{3} - 0) \right] \]
\[ = \left( \frac{1}{\sqrt{3}} \right) (\sqrt{3} - \sqrt{3}) \]
\[ = 0 \]

(b) \( h(x) = -|x| \) on \( \text{(i.) } [-1, 0], \text{ (ii.) } [0, 1], \text{ (iii.) } [-1, 1] \).
i.

\[ av(f) = \left( \frac{1}{0 - (-1)} \right) \int_{-1}^{0} -|x| \, dx \]

\[ = (1) \int_{-1}^{0} (-x) \, dx \]

\[ = \int_{-1}^{0} x \, dx \]

\[ = \frac{1}{2} x^2 \bigg|_{-1}^{0} \]

\[ = 0 - \frac{1}{2} \]

\[ = -\frac{1}{2} \]

ii.

\[ av(f) = \left( \frac{1}{1 - 0} \right) \int_{0}^{1} -|x| \, dx \]

\[ = (1) \int_{0}^{1} -x \, dx \]

\[ = -\frac{1}{2} x^2 \bigg|_{0}^{1} \]

\[ = -\frac{1}{2} \]
\[ av(f) = \left( \frac{1}{1 - (-1)} \right) \int_{-1}^{1} -|x| \, dx \]
\[ = \left( \frac{1}{2} \right) \left[ \int_{-1}^{0} -|x| \, dx + \int_{0}^{1} -|x| \, dx \right] \]
\[ = \left( \frac{1}{2} \right) \left[ \int_{-1}^{0} (-x) \, dx + \int_{0}^{1} -x \, dx \right] \]
\[ = \left( \frac{1}{2} \right) \left[ \frac{1}{2} x^2 \bigg|_{-1}^{0} - \frac{1}{2} x^2 \bigg|_{0}^{1} \right] \]
\[ = \left( \frac{1}{2} \right) \left[ (0 - \left( \frac{1}{2} \right)) - (\frac{1}{2} - 0) \right] \]
\[ = \left( \frac{1}{2} \right) \left[ -\frac{1}{2} - \frac{1}{2} \right] \]
\[ = \left( \frac{1}{2} \right) (-1) \]
\[ = -\frac{1}{2} \]

7. (a) Consider the partition \( P \) that subdivides the interval \([a, b]\) into \( n \) subintervals of width \( \Delta x = \frac{b - a}{n} \) and let \( c_k \) be the right endpoint of each subinterval. So the partition is
\[ P = \{ a, a + \frac{b - a}{n}, a + \frac{2(b - a)}{n}, \ldots, a + \frac{n(b - a)}{n} = b \} \]
and \( c_k = a + \frac{k(b-a)}{n} \). We get the Riemann sum
\[
\sum_{k=1}^{n} f(c_k) \Delta x = \sum_{k=1}^{n} c_k^2 \left( \frac{b-a}{n} \right) \\
= \frac{b-a}{n} \sum_{k=1}^{n} \left( a + \frac{k(b-a)}{n} \right) \\
= \frac{b-a}{n} \sum_{k=1}^{n} \left( a^2 + \frac{2ak(b-a)}{n} + \frac{k^2(b-a)^2}{n^2} \right) \\
= \frac{b-a}{n} \left( \sum_{k=1}^{n} a^2 + \frac{2a(b-a)}{n} \sum_{k=1}^{n} k + \frac{(b-a)^2}{n^2} \sum_{k=1}^{n} k^2 \right) \\
= \frac{b-a}{n} \cdot na^2 + \frac{2a(b-a)^2}{n} \cdot \frac{n(n+1)}{2} + \frac{(b-a)^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \\
= (b-a)a^2 + a(b-a)^2 \cdot \frac{n+1}{n} + \frac{(b-a)^3(2n^2 + 3n + 1)}{6n^2} \\
= (b-a)a^2 + a(b-a)^2 \cdot \left( 1 + \frac{1}{n} \right) + (b-a)^3 \left( \frac{2 + \frac{3}{n} + \frac{1}{n^2}}{6} \right).
\]

As \( n \to \infty \) and \( ||P|| \to 0 \), this expression converges to
\[
(b-a)a^2 + a(b-a)^2 + (b-a)^3 \cdot \frac{2}{6},
\]
which simplifies to \( \frac{b^3}{3} - a^3 \). Thus \( \int_{a}^{b} x^2 \, dx = \frac{b^3}{3} - \frac{a^3}{3} \).

(b) Consider the partition \( P \) that subdivides the interval \([-1, 0]\) into \( n \) subintervals of width \( \Delta x = \frac{1}{n} \) and let \( c_k \) be the right endpoint of each subinterval. So the partition is
\[
P = \{-1, -1 + \frac{1}{n}, -1 + \frac{2}{n}, \ldots, -1 + \frac{n}{n} = 0\}
\]
and \( c_k = -1 + \frac{k}{n} \). We get the Riemann sum
\[
\sum_{k=1}^{n} f(c_k) \Delta x = \sum_{k=1}^{n} \left( (-1 + \frac{k}{n}) - (-1 + \frac{k}{n})^2 \right) \cdot \frac{1}{n} \\
= \frac{1}{n} \sum_{k=1}^{n} \left( -1 + \frac{k}{n} - 1 + \frac{2k}{n} - \left( \frac{k}{n} \right)^2 \right) \\
= \frac{-2}{n} \sum_{k=1}^{n} 1 + \frac{3}{n^2} \sum_{k=1}^{n} k - \frac{1}{n^3} \sum_{k=1}^{n} k^2 \\
= \frac{-2}{n} + \frac{3}{n^2} \left( \frac{n(n+1)}{2} \right) - \frac{1}{n^3} \left( \frac{n(n+1)(2n+1)}{6} \right) \\
= -2 + \frac{3}{2} \left( 1 + \frac{1}{n} \right) + \frac{1}{n^3} \left( \frac{2n^3 + 3n^2 + n}{6n^2} \right).
\]
As $n \to \infty$ and $||P|| \to 0$, this expression converges to $-2 + \frac{3}{2} - \frac{2}{6} = -\frac{5}{6}$. Thus $\int_{-1}^{0} (x - x^2) \, dx = -\frac{5}{6}$.

8. Use the Max-Min Inequality to find upper and lower bounds for the value of

$$\int_{0}^{1} \frac{1}{1 + x^2} \, dx$$

Notice that the integrand $f(x) = \frac{1}{1 + x^2}$ is a decreasing function ($f'(x) = \frac{-2x}{(1 + x^2)^2} \leq 0$ for $x \in [0, 1]$). Therefore, on the interval $f$ has a maximum at $x = 0$ and has a minimum at $x = 1$.

$$\max f = f(0) = 1 \quad \min f = f(1) = \frac{1}{2}$$

Thus the Max-Min Inequality gives:

$$\frac{1}{2} \leq \int_{0}^{1} \frac{1}{1 + x^2} \, dx \leq 1$$

9. Use the Max-Min Inequality to find upper and lower bounds for the value of

$$\int_{0}^{0.5} \frac{1}{1 + x^2} \, dx \quad \text{and} \quad \int_{0.5}^{1} \frac{1}{1 + x^2} \, dx$$

Using the observation (from 65) that $f(x) = \frac{1}{1 + x^2}$ is decreasing on $[0, 0.5]$ and $[0.5, 1]$, we get the following two inequalities:

$$0.4 \leq \int_{0}^{0.5} \frac{1}{1 + x^2} \, dx \leq 0.5$$

$$0.25 \leq \int_{0.5}^{1} \frac{1}{1 + x^2} \, dx \leq 0.4.$$ 

Therefore, by adding the two inequalities we get the new (and improved) estimate:

$$0.65 \leq \int_{0}^{1} \frac{1}{1 + x^2} \, dx \leq 0.9.$$ 

10. We know that $\sin x \leq x$ for $x \geq 0$. Therefore we can get the upper bound:
\[
\int_0^1 \sin x \, dx \leq \int_0^1 x \, dx = \frac{1}{2} \cdot 1^2 - \frac{1}{2} \cdot 0^2 = \frac{1}{2}.
\]

Section 5.4:

1. (a) \[
\int_{-2}^0 (2x + 5) \, dx = (x^2 + 5x) \bigg|_{-2}^0 = (0 + 0) - (4 - 10) = 6
\]

(b) \[
\int_{-3}^4 \left(5 - \frac{x^2}{4}\right) \, dx = \left(5x - \frac{x^3}{4}\right) \bigg|_{-3}^4 = (20 - 4) - \left(-15 - \frac{9}{4}\right) = 16 + \frac{63}{4} = \frac{73}{4}
\]

(c) \[
\int_0^\pi \sin x \, dx = \left[-\cos x\right]_0^\pi = -\cos(\pi) + \cos(0) = 1 + 1 = 2
\]

(d) \[
\int_0^{\pi/3} 2 \sec^2 x \, dx = 2 \tan \left(\frac{\pi}{3}\right) - 2 \tan(0) = 2 \cdot \sqrt{3} - 2 \cdot 0 = 2\sqrt{3}
\]

(e) \[
\int_1^{-1} (r+1)^2 \, dr = -\int_1^{-1} (r^2 + 2r + 1) \, dr = -\left[\left(\frac{1}{3} r^3 + r^2 + r\right)\right]_{-1}^1 = -\left[\frac{1}{3} + 1 + 1\right] - \left(-\frac{1}{3} + 1 - 1\right) = \left[-\frac{7}{3} + \frac{1}{3}\right] = \frac{8}{3}
\]

(f) \[
\int_1^2 \left(\frac{1}{x} - e^{-x}\right) \, dx = \left(\ln x + e^{-x}\right) \bigg|_1^2 = (\ln 2 + e^{-2}) - (0 + e^{-1})
\]

(g) \[
\int_0^1 \frac{4}{1 + x^2} \, dx = 4 \arctan x \bigg|_0^1 = 4 \arctan(1) - 4 \arctan(0) = 4 \cdot \frac{\pi}{4} - 4 \cdot 0 = \pi - 4
\]

(h) \[
\int_2^5 \frac{x}{\sqrt{1 + x^2}} \, dx = \int_2^5 x (1+x^2)^{-1/2} \, dx = \frac{1}{2} \cdot 2 \cdot \sqrt{1 + x^2} \bigg|_2^5 = \sqrt{1 + x^2} \bigg|_2^5 = \sqrt{26} - \sqrt{5}
\]
(i) \( \int_{0}^{1} xe^{x^2} \, dx \)

Notice that \( \frac{d}{dx} x^2 = 2x \), and we have an \( x^2 \) in the exponent and an \( x \) in the product. Thus, keeping in mind the chain rule we get:

\[
f(x) = xe^{x^2} \quad \Rightarrow \quad F(x) = \frac{1}{2}e^{x^2}
\]

\[
\int_{0}^{1} xe^{x^2} \, dx = F(1) - F(0) = \frac{1}{2}e - \frac{1}{2} = \frac{1}{2}(e - 1).
\]

(j) \( \int_{1}^{0} \frac{\ln x}{x} \, dx \)

Notice that \( \frac{d}{dx} \ln x = \frac{1}{x} \), and we have an \( \ln x \) and a \( \frac{1}{x} \). Thus, keeping in mind the chain rule we get:

\[
f(x) = \frac{\ln x}{x} \quad \Rightarrow \quad F(x) = \frac{1}{2} (\ln x)^2
\]

\[
\int_{1}^{0} \frac{\ln x}{x} \, dx = F(2) - F(1) = \frac{1}{2} (\ln 2)^2 - 0 = \frac{1}{2} (\ln 2)^2.
\]

2. (a) \( \frac{d}{dx} \int_{0}^{\sqrt{x}} \cos t \, dt \)

i. \[
\int_{0}^{\sqrt{x}} \cos t \, dt = \sin t \big|_{0}^{\sqrt{x}} = \sin(\sqrt{x}) - \sin(0) = \sin(\sqrt{x})
\]

Therefore we get:

\[
\frac{d}{dx} \int_{0}^{\sqrt{x}} \cos t \, dt = \frac{d}{dx} \sin(\sqrt{x}) = \cos(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}
\]

ii. Let \( u = \sqrt{x} \)

\[
\frac{d}{dx} \int_{0}^{\sqrt{x}} \cos t \, dt = \left( \frac{d}{du} \int_{0}^{u} \cos t \, dt \right) \cdot \frac{du}{dx}
\]

\[
= \cos u \cdot \frac{du}{dx}
\]

\[
= \cos(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}
\]
(b) \( \frac{d}{d\theta} \int_{0}^{\tan\theta} \sec^2 y \, dy \)

i. \[ \int_{0}^{\tan\theta} \sec^2 y \, dy = \tan y \bigg|_{0}^{\tan\theta} = \tan(\tan\theta) - \tan(0) = \tan(\tan\theta) \]

Therefore we get:
\[ \frac{d}{d\theta} \int_{0}^{\tan\theta} \sec^2 y \, dy = \frac{d}{d\theta} \tan(\tan\theta) = \sec^2(\tan\theta) \cdot \sec^2 \theta \]

ii. Let \( u = \tan\theta \)
\[ \frac{d}{d\theta} \int_{0}^{\tan\theta} \sec^2 y \, dy = \left( \frac{d}{du} \int_{0}^{u} \sec^2 y \, dy \right) \cdot \frac{du}{d\theta} = \sec^2 u \cdot \frac{du}{d\theta} = \sec^2(\tan\theta) \cdot \sec^2 \theta \]

3. \( y = \int_{\sqrt[3]{x}}^{0} \sin(t^2) \, dt = -\int_{0}^{\sqrt[3]{x}} \sin(t^2) \, dt \)

Let \( u = \sqrt[3]{x} \). Then we get:
\[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\sin(u^2) \cdot \frac{1}{x^{\frac{1}{3}}} = -\sin(x) \cdot \frac{1}{x^{\frac{1}{3}}} = \frac{\sin(x)}{2\sqrt{x}} \]

4. (a) \( y = -x^2 - 2x \) where \(-3 \leq x \leq 2\)

First, notice that \( y = -x^2 - 2x = -x(x+2) \). Therefore in the interval \([-3, 2]\), the function has zeroes at \( x = 0 \) and \( x = -2 \).

Area = \[-\int_{-3}^{-2} (-x^2 - 2x) \, dx + \int_{-2}^{0} (-x^2 - 2x) \, dx - \int_{0}^{2} (-x^2 - 2x) \, dx \]
\[ = -\left( -\frac{1}{3}x^3 - x^2 \right) \bigg|_{-3}^{-2} + \left( -\frac{1}{3}x^3 - x^2 \right) \bigg|_{-2}^{0} - \left( -\frac{1}{3}x^3 - x^2 \right) \bigg|_{0}^{2} \]
\[ = -\left[ \left( \frac{8}{3} - 4 \right) - (9 - 9) \right] \, + \, \left[ (0 - 0) \, - \, \left( \frac{8}{3} - 4 \right) \right] \, - \, \left[ \left( -\frac{8}{3} - 4 \right) \, - \, (0 - 0) \right] \]
\[ = \frac{4}{3} + 4 + \frac{20}{3} \]
\[ = \frac{28}{3} \]
(b) \( y = 3x^2 - 3 \) where \(-2 \leq x \leq 2\)

First notice that \( y = 3(x^2 - 1) = 3(x - 1)(x + 1)\). Therefore in the interval \([-2, 2]\), the function has zeroes at \( x = \pm 1\).

\[
\text{Area} = \int_{-2}^{-1} (3x^2 - 3) \, dx - \int_{-1}^{1} (3x^2 - 2) \, dx + \int_{1}^{2} (3x^2 - 3) \, dx
\]

\[
= \left( x^3 - 3x \right) \bigg|_{-2}^{-1} - \left( x^3 - 3x \right) \bigg|_{-1}^{1} + \left( x^3 - 3x \right) \bigg|_{1}^{2}
\]

\[
= [(-1 + 3) - (-8 + 6)] - [(1 - 3) - (1 + 3)] + [(8 - 6) - (1 - 3)]
\]

\[
= 4 + 4 + 4
\]

\[
= 12
\]

(c) \( y = x^{1/3} - x \) where \(-1 \leq x \leq 8\)

First notice that \( y = x^{1/3} - x = x^{1/3} \left( 1 - x^{1/3} \right) \left( 1 + x^{1/3} \right)\), so the on \([-1, 8]\) the function has zeroes at \( x = 0, \pm 1\).

\[
\text{Area} = -\int_{-1}^{0} (x^{1/3} - x) \, dx + \int_{0}^{1} (x^{1/3} - x) \, dx - \int_{1}^{8} (x^{1/3} - x) \, dx
\]

\[
= -\left( \frac{3}{4}x^{4/3} - \frac{1}{2}x^2 \right) \bigg|_{-1}^{0} + \left( \frac{3}{4}x^{4/3} - \frac{1}{2}x^2 \right) \bigg|_{0}^{1} - \left( \frac{3}{4}x^{4/3} - \frac{1}{2}x^2 \right) \bigg|_{1}^{8}
\]

\[
= -\left[ (0 - 0) - \left( \frac{3}{4} - \frac{1}{2} \right) \right] + \left[ \left( \frac{3}{4} - \frac{1}{2} \right) - (0 - 0) \right] - \left[ (12 - 32) - \left( \frac{3}{4} - \frac{1}{2} \right) \right]
\]

\[
= \frac{1}{4} + \frac{1}{4} + \frac{81}{4}
\]

\[
= \frac{83}{4}
\]

5. (a) \( v = \frac{ds}{dt} = \frac{df}{dt} \int_{0}^{t} f(x) \, dx = f(t) \).

Therefore \( v(5) = f(5) = 2 \) m/s.

(b) Recall that the acceleration is the rate of change of the velocity, so \( a = \frac{df}{dt} \). Since \( f \) has a negative slope at \( t = 5 \), the acceleration is negative at \( t = 5 \).

(c) Since \( s = \int_{0}^{t} f(x) \, dx \), we know that the position at \( t = 3 \) can be represented as the area under the graph on \([0, 3]\). Notice that on this interval, \( f \) is a straight line and so the value of the integral is just the area of a triangle formed by \( f(x) \), the \( x \)-axis and \( x = 3 \).
Therefore \( s(3) = \frac{1}{2} \cdot 3 \cdot 3 = \frac{9}{2} \)

(d) We want to know when \( s \) has a maximum and we will determine this by looking at \( f(x) \), which is the derivative of the \( s \). We know \( f \) is positive on \([0, 6] \) and negative on \([6, 9] \), which means that \( s \) goes from increasing to decreasing at \( t = 6 \). By the first derivative test, we know that \( s \) must have relative max at \( t = 6 \).

(e) At \( x \approx 4 \) and \( x \approx 7 \) since \( s''(x) = f'(x) \) is zero at these points.

(f) The particle is moving towards the origin when it has negative velocity \( \rightarrow f \) is negative. Hence it is moving towards the origin on \([6, 9] \). Similarly the particle is moving away from the origin when \( f \) is positive, hence on \([0, 6] \).

(g) Judging by the graph of \( f \), we know that the particle is to the right of the origin (on the positive side) because \( s(9) = \int_0^9 f(x) \, dx > 0 \) (the area above the \( x \)-axis is greater than the area below it).

6. Find \( \lim_{x \to 0} \frac{1}{x^3} \int_0^x \frac{t^2}{t^4 + 1} \, dt \)

Notice that \( \lim_{x \to 0} \int_0^x \frac{t^2}{t^4 + 1} \, dt = 0 \) by the rules of integrals and \( \lim_{x \to 0} x^3 = 0 \). Thus if you think of the expression inside the limit as a fraction, we have that both the numerator and the denominator go to \( 0 \). Hence we can apply l’Hôpital’s Rule to the limit and get:

\[
\lim_{x \to 0} \frac{1}{x^3} \int_0^x \frac{t^2}{t^4 + 1} \, dt = \lim_{x \to 0} \left[ \left( \int_0^x \frac{t^2}{t^4 + 1} \, dt \right) / x^3 \right] \\
= \lim_{x \to 0} \left[ \left( \frac{x^2}{x^4 + 1} \right) / (3x^2) \right] \\
= \lim_{x \to 0} \frac{1}{3(x^4 + 1)} \\
= \frac{1}{3}
\]

7. Suppose \( f'(x) \geq 0 \) for all values of \( x \), and that \( f(1) = 0 \). Define:

\[
g(x) = \int_0^x f(t) \, dt
\]

(a) True. It follows from the FTC, part I.

(b) True. Differentiable \( \Rightarrow \) Continuous.
(c) True. We know that \( g'(1) = f(1) = 0 \), so the tangent line at \( x = 0 \) is horizontal.

(d) False. We know that \( f \) crosses the \( x \)-axis at \( x = 1 \) and that \( f \) is always increasing. Therefore \( g'(x) = f(x) \) is negative for \( x < 1 \) and \( g'(x) = f(x) \) is positive for \( x > 1 \). By the first derivative test, this is a minimum, not a max.

Note that you could also use the fact that \( g'(1) = 0 \) and \( g''(1) > 0 \)

(e) True. See (d.)

(f) False. \( g''(x) = f'(x) > 0 \), and hence does not change sign.

(g) True. \( g'(x) = f(x) \) and we know that the \( f(1) = 0 \) and is increasing.