Math 21B-B - Homework Set 3

Section 4.8:

1. (a) i. $f(x) = 2x^{-3}$
   $F(x) = -x^{-2}$

   ii. $f(x) = \frac{x^{-3}}{2} + x^2$
   $F(x) = -\frac{x^{-2}}{4} + \frac{1}{3}x^3$

   iii. $f(x) = -x^{-3} + x - 1$
   $F(x) = \frac{1}{2}x^{-2} + \frac{1}{2}x^2 - x$

   (b) i. $f(x) = \frac{1}{2}x^{-1/2}$
   $F(x) = x^{1/2}$

   ii. $f(x) = -\frac{1}{2}x^{-3/2}$
   $F(x) = x^{-1/2}$

   iii. $f(x) = -\frac{3}{2}x^{-5/2}$
   $F(x) = x^{-3/2}$

   (c) i. $f(x) = -\pi \sin(\pi x)$
   $F(x) = \cos(\pi x)$

   ii. $f(x) = 3\sin(x)$
   $F(x) = -3\cos(x)$

   iii. $f(x) = \sin(\pi x) - 3\sin(3x)$
   $F(x) = -\frac{1}{\pi}\cos(\pi x) + \cos(3x)$

   (d) i. $f(x) = \sec^2(x)$
   $F(x) = \tan(x)$

   ii. $f(x) = \frac{2}{3}\sec^2\left(\frac{x}{3}\right)$
   $F(x) = 2\tan\left(\frac{x}{3}\right)$

   iii. $f(x) = -\sec^2\left(\frac{3x}{2}\right)$
   $F(x) = -\frac{2}{3}\tan\left(\frac{3x}{2}\right)$

   (e) i. $f(x) = \sec(x)\tan(x)$
   $F(x) = \sec(x)$

   ii. $f(x) = 4\sec(3x)\tan(3x)$

$F(x) = \frac{4}{3}\sec(3x)$

iii. $f(x) = \sec\left(\frac{\pi x}{2}\right)\tan\left(\frac{\pi x}{2}\right)$
$F(x) = \frac{2}{\pi}\sec\left(\frac{\pi x}{2}\right)$

(f) i. $f(x) = e^{3x}$
$F(x) = \frac{1}{3}e^{3x}$

ii. $f(x) = e^{-x}$
$F(x) = -e^{-x}$

iii. $f(x) = e^{x/2}$
$F(x) = 2e^{x/2}$

2. (a) $\displaystyle\int \left(3t^2 + \frac{t}{2}\right)\, dt = t^3 + \frac{t^2}{4} + C$

$\frac{d}{dt}\left(t^3 + \frac{t^2}{4} + C\right) = 3t^2 + 2\cdot\frac{t}{4} + 0 = 3t^2 + \frac{t}{2}$

(b) $\displaystyle\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right)\, dx = \frac{x^{3/2}}{3} + 4\sqrt{x} + C$

$\frac{d}{dx}\left(\frac{x^{3/2}}{3} + 4\sqrt{x} + C\right) = \frac{3}{2}\cdot\frac{x^{1/2}}{3} + \frac{1}{2}\cdot 4x^{-1/2} + 0 = \frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}$

(c) $\displaystyle\int \left(7\sin\left(\frac{\theta}{3}\right)\right)\, d\theta = -21\cos\left(\frac{\theta}{3}\right) + C$

$\frac{d}{d\theta}\left(-21\cos\left(\frac{\theta}{3}\right) + C\right) = \frac{1}{3}\cdot 21\sin\left(\frac{\theta}{3}\right) + 0 = 7\sin\left(\frac{\theta}{3}\right)$

(d) $\displaystyle\int \left(2e^x - 3e^{-2x}\right)\, dx = 2e^x + \frac{3}{2}e^{-2x} + C$

$\frac{d}{dx}\left(2e^x + \frac{3}{2}e^{-2x} + C\right) = 2e^x + (-2)\cdot\frac{3}{2}e^{-2x} + 0 = 2e^x - 3e^{-2x}$

(e) $\displaystyle\int \left(1 + \tan^2\theta\right)\, d\theta = \int \sec^2\theta\, d\theta = \tan\theta + C$

$\frac{d}{d\theta}(\tan\theta + C) = \sec^2\theta + 0 = 1 + \tan^2\theta$

(f) $\displaystyle\int \left(2 + \tan^2\theta\right)\, d\theta = \int \left(1 + \sec^2\theta\right)\, d\theta = \theta + \tan\theta + C$

$\frac{d}{d\theta}(\theta + \tan\theta + C) = 1 + \sec^2\theta + 0 = 1 + 1 + \tan^2\theta = 2 + \tan^2\theta$

3. (a)

$$\begin{aligned}\frac{d}{dx}\left(-\frac{(3x+5)^{-1}}{3} + C\right) &= \frac{d}{dx}\left(-\frac{1}{3}(3x+5)^{-1} + C\right)\\ &= -\frac{1}{3}\cdot -(3x+5)^{-2}(3)\\ &= (3x+5)^{-2}\end{aligned}$$

(b)

$$\frac{d}{dx}\left(-\frac{1}{x+1}+C\right) = \frac{d}{dx}\left[-(x+1)^{-1}+C\right]$$
$$= (x+1)^{-2}$$
$$= \frac{1}{(x+1)^2}$$

4. (a) Wrong.

   (b) Wrong.

   (c) Right.

5. (a) Wrong.

   (b) Right.

   (c) Right.

6. Suppose that $f(x) = \frac{d}{dx}(1-\sqrt{x})$ and $g(x) = \frac{d}{dx}(x+2)$.

   (a) $\displaystyle\int f(x)\,dx = 1-\sqrt{x}+C = -\sqrt{x}+C'$

   (b) $\displaystyle\int g(x)\,dx = x+2+C = x+C'$

   (c) $\displaystyle\int [-f(x)]\,dx = -\int f(x)\,dx = -1+\sqrt{x}+C = \sqrt{x}+C'$

   (d) $\displaystyle\int [-g(x)]\,dx = -\int g(x)\,dx = -x-2+C = -x+C'$

   (e) $\displaystyle\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx = 1-\sqrt{x}+x+2+C = -\sqrt{x}+x+C'$

   (f) $\displaystyle\int [f(x)-g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx = 1-\sqrt{x}-x-2+C = -\sqrt{x}-x+C'$

Section 5.5:

1. (a) $\displaystyle\int x\sin\left(2x^2\right)\,dx.$

   Let $u = 2x^2$. Then $du = 4x\,dx \quad \Rightarrow \quad dx = \frac{1}{4x}\,du.$

$$\begin{aligned}\int x\sin\left(2x^2\right)\,dx &= \int \frac{1}{4}\sin(u)\,du\\ &= -\frac{1}{4}\cos(u)+C\\ &= -\frac{1}{4}\cos\left(2x^2\right)+C.\end{aligned}$$

(b) $\int 28(7x-2)^{-5}\,dx$.

Let $u = 7x-2$. The $du = 7\,dx \quad\Rightarrow\quad dx = \frac{1}{7}\,du$.

$$\begin{aligned}\int 28(7x-2)^{-5}\,dx &= \int 4u^{-5}\,du\\ &= -u^{-4}+C\\ &= -(7x-2)^{-4}+C\end{aligned}$$

(c) $\int \frac{9r^2\,dr}{\sqrt{1-r^3}}$.

Let $u = 1-r^3$. Then $du = -3r^2\,dr \quad\Rightarrow\quad dr = -\frac{1}{3r^2}\,du$.

$$\begin{aligned}\int \frac{9r^2\,dr}{\sqrt{1-r^3}} &= \int -\frac{3\,du}{\sqrt{u}}\\ &= \int -3u^{-1/2}\,du\\ &= -6u^{1/2}+C\\ &= -6\sqrt{1-r^3}+C\end{aligned}$$

(d) $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)\,dx$.

Let $u = \frac{1}{x}$. Then $du = -\frac{1}{x^2}\,dx \quad\Rightarrow\quad dx = -x^2\,du$.

$$\begin{aligned}\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)\,dx &= \int -\cos^2 u\,du\\ &= \int -\left[\frac{1}{2}+\frac{\cos(2u)}{2}\right]\,du \qquad \text{(Half-Angle Formula)}\\ &= -\frac{u}{2}-\frac{\sin(2u)}{4}+C\\ &= -\frac{1}{2x}-\frac{1}{4}\sin\left(\frac{2}{x}\right)+C.\end{aligned}$$

(e) $\int \sqrt{3-2s}\, ds.$

Let $u = 3 - 2s$. Then $du = -2\, ds \quad \Rightarrow \quad ds = -\frac{1}{2}\, du$

$$\begin{aligned} \int \sqrt{3-2s}\, ds &= \int -\frac{1}{2}\sqrt{u}\, du \\ &= -\frac{1}{3}u^{3/2} + C \\ &= -\frac{1}{3}(3-2s)^{3/2} + C. \end{aligned}$$

2. (a) $\int \frac{4y\, dy}{\sqrt{2y^2+1}}.$

Let $u = 2y^2 + 1$. Then $du = 4y\, dy \quad \Rightarrow \quad dy = \frac{1}{4y}\, du.$

$$\begin{aligned} \int \frac{4y\, dy}{\sqrt{2y^2+1}} &= \int \frac{1}{\sqrt{u}}\, du \\ &= 2\sqrt{u} + C \\ &= 2\sqrt{2y^2+1} + C \end{aligned}$$

(b) $\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx.$

Let $u = \cos x$. Then $du = -\sin x\, dx \quad \Rightarrow \quad dx = -\frac{1}{\sin x}\, du.$

$$\begin{aligned} \int \frac{\sin x}{\cos x}\, dx &= \int -\frac{1}{u}\, du \\ &= -\ln|u| + C \\ &= -\ln|\cos x| + C \\ &= \ln|\sec x| + C. \qquad \text{(Properties of ln)} \end{aligned}$$

(c) $\int x^{1/3} \sin\left(x^{4/3} - 8\right)\, dx.$

Let $u = x^{4/3} - 8$. Then $du = \frac{4}{3}x^{1/3}\, dx \quad \Rightarrow \quad dx = \frac{3}{4}x^{-1/3}\, du.$

$$\begin{aligned} \int x^{1/3} \sin\left(x^{4/3} - 8\right)\, dx &= \int \frac{3}{4}\sin u\, du \\ &= -\frac{3}{4}\cos u + C \\ &= -\frac{3}{4}\cos\left(x^{4/3} - 8\right) + C. \end{aligned}$$

(d) $\int \sqrt{\frac{x-1}{x^5}}\,dx = \int \frac{1}{x^2}\sqrt{\frac{x-1}{x}}\,dx$

Let $u = \frac{x-1}{x}$. Then $du = \frac{1}{x^2}\,dx \quad \Rightarrow \quad dx = x^2\,du$

$$\begin{aligned}\int \frac{1}{x^2}\sqrt{\frac{x-1}{x}}\,dx &= \int \sqrt{u}\,du \\ &= \frac{2}{3}u^{3/2} + C \\ &= \frac{2}{3}\left(\frac{x-1}{x}\right)^{3/2} + C\end{aligned}$$

(e) $\int (\cos x)e^{\sin x}\,dx$

Let $u = \sin x$. Then $du = \cos x\,dx \quad \Rightarrow \quad dx = \frac{du}{\cos x}$

$$\begin{aligned}\int (\cos x)e^{\sin x}\,dx &= \int e^u\,du \\ &= e^u + C \\ &= e^{\sin x} + C\end{aligned}$$

(f) $\int \frac{1}{x^2}\,e^{1/x}\sec\left(1+e^{1/x}\right)\tan\left(1+e^{1/x}\right)\,dx$

Let $u = 1+e^{1/x}$. Then $du = e^{1/x}\cdot -\frac{1}{x^2}\,dx \quad \Rightarrow \quad dx = -x^2e^{-1/x}\,du$

$$\begin{aligned}\int \frac{1}{x^2}\,e^{1/x}\sec\left(1+e^{1/x}\right)\tan\left(1+e^{1/x}\right)\,dx &= \int -\sec u\tan u\,du \\ &= -\sec u + C \\ &= -\sec\left(1+e^{1/x}\right) + C\end{aligned}$$

(g) $\int \frac{dx}{x\ln x}$

Let $u = \ln x$. Then $du = \frac{1}{x}\,dx \quad \Rightarrow \quad dx = x\,du$

$$\begin{aligned}\int \frac{dx}{x\ln x} &= \int \frac{du}{u} \\ &= \ln u + C \\ &= \ln|\ln x| + C\end{aligned}$$

(h) $\displaystyle\int \frac{1}{\sqrt{e^{2\theta}-1}}\,d\theta = \int \frac{e^\theta}{e^\theta\sqrt{e^{2\theta}-1}}\,d\theta$

Let $u = e^\theta$. Then $du = e^\theta\,d\theta \quad \Rightarrow \quad d\theta = e^{-\theta}\,du$

$$\begin{aligned}\int \frac{e^\theta}{e^\theta\sqrt{e^{2\theta}-1}}\,d\theta &= \int \frac{du}{u\sqrt{u^2-1}}\,du \\ &= \sec^{-1} u + C \\ &= \sec^{-1}\left(e^\theta\right) + C\end{aligned}$$

(i) $\displaystyle\int \frac{18\tan^2 x\sec^2 x}{\left(2+\tan^3 x\right)^2}\,dx$

Three ways:

a. $u = \tan x, \quad du = \sec^2 x\,dx$

$$\int \frac{18\tan^2 x\sec^2 x}{\left(2+\tan^3 x\right)^2}\,dx = \int \frac{18u^2}{(2+u^3)^2}\,du$$

$v = u^3, \quad dv = 3u^2\,du$

$$\int \frac{18u^2}{(2+u^3)^2}\,du = \int \frac{6}{(2+v)^2}\,dv$$

$w = 2+v, \quad dw = dv$

$$\begin{aligned}\int \frac{6}{(2+v)^2}\,dv &= \int \frac{6}{w^2}\,dw \\ &= -\frac{6}{w} + C \\ &= -\frac{6}{2+v} + C \\ &= -\frac{6}{2+u^3} + C \\ &= -\frac{6}{2+\tan^3 x} + C\end{aligned}$$

b. $u = \tan^3 x, \quad du = 3\tan^2 x \cdot \sec^2 x\,du$

$$\int \frac{18\tan^2 x\sec^2 x}{\left(2+\tan^3 x\right)^2}\,dx = \int \frac{6}{(2+u)^2}\,du$$

$v = 2+u, \quad dv = du$

$$\int \frac{6}{(2+u)^2}\,du = \int \frac{6}{v^2}\,dv$$
$$= -\frac{6}{v} + C$$
$$= -\frac{6}{2+u} + C$$
$$= -\frac{6}{2+\tan^3 x} + C$$

c. $u = 2 + \tan^3 x, \quad du = 3\tan^2 x \cdot \sec^2 x\, dx$

$$\int \frac{18\tan^2 x \sec^2 x}{\left(2+\tan^3 x\right)^2}\,dx = \int \frac{6}{u^2}\,du$$
$$= -\frac{6}{u} + C$$
$$= -\frac{6}{2+\tan^3 x} + C$$

3. (a) $\frac{ds}{dt} = 12t\left(3t^2-1\right)^3, \quad s(1) = 3$

$$\begin{aligned} s(t) &= \int 12t\left(3t^2-1\right)^3\,dt \\ &= \int 2u^3\,du \qquad\qquad (u = 3t^2 - 1, \quad du = 6t\,dt) \\ &= \frac{1}{2}u^4 + C \\ &= \frac{1}{2}\left(3t^2-1\right)^4 + C \end{aligned}$$

We now use the initial condition $s(1) = 3$ to find the value of C.

$$\begin{aligned} 3 = \frac{1}{2}\left(3(1)^2-1\right)^4 + C \quad &\Leftrightarrow \quad 3 = \frac{1}{2}\cdot 2^4 + C \\ &\Leftrightarrow \quad 3 = 8 + C \\ &\Leftrightarrow \quad -5 = C \end{aligned}$$

Therefore we get:

$$s(t) = \frac{1}{2}\left(3t^2-1\right)^4 - 5$$

(b) $\frac{dr}{d\theta} = 3\cos^2\left(\frac{\pi}{4} - \theta\right), \quad r(0) = \frac{\pi}{8}$

$$\begin{aligned}
r(\theta) &= \int 3\cos^2\left(\frac{\pi}{4} - \theta\right)\, d\theta \\
&= \int 3\left[\frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{2} - 2\theta\right)\right]\, d\theta \\
&= \int -\frac{3}{2}\left[\frac{1}{2} + \frac{1}{2}\cos u\right]\, du \qquad \left(u = \tfrac{\pi}{2} - 2\theta, \quad du = -2\, d\theta\right) \\
&= -\frac{3}{2}\left[\frac{u}{2} + \frac{1}{2}\sin u\right] + C \\
&= -\frac{3}{2}\left[\frac{\pi/2 - 2\theta}{2} + \frac{1}{2}\sin\left(\frac{\pi}{2} - 2\theta\right)\right] + C
\end{aligned}$$

We now use the initial condition $r(0) = \frac{\pi}{8}$ to find the value of C.

$$\begin{aligned}
\frac{\pi}{8} = -\frac{3}{2}\left[\frac{\pi/2}{2} + \frac{1}{2}\sin\left(\frac{\pi}{2}\right)\right] + C \quad &\Leftrightarrow \quad \frac{\pi}{8} = -\frac{3}{2}\left[\frac{\pi}{4} + \frac{1}{2}\right] + C \\
&\Leftrightarrow \quad \frac{\pi}{8} = -\frac{3\pi}{8} - \frac{3}{4} + C \\
&\Leftrightarrow \quad \frac{\pi}{2} + \frac{3}{4} = C
\end{aligned}$$

Therefore we get:

$$\begin{aligned}
r(\theta) &= -\frac{3}{2}\left[\frac{\pi/2 - 2\theta}{2} + \frac{1}{2}\sin\left(\frac{\pi}{2} - 2\theta\right)\right] + \frac{\pi}{2} + \frac{3}{4} \\
&= -\frac{3\pi}{8} + \frac{3\theta}{2} - \frac{3}{4}\sin\left(\frac{\pi}{2} - 2\theta\right) + \frac{\pi}{2} + \frac{3}{4} \\
&= \frac{\pi}{8} + \frac{3\theta}{2} - \frac{3}{4}\sin\left(\frac{\pi}{2} - 2\theta\right) + \frac{3}{4} \\
&= \frac{\pi}{8} + \frac{3\theta}{2} - \frac{3}{4}\cos(2\theta) + \frac{3}{4}
\end{aligned}$$

(c) $\frac{d^2s}{dt^2} = -4\sin\left(2t - \frac{\pi}{2}\right), \quad s'(0) = 100, \; s(0) = 0$

$$\begin{aligned}
s'(t) &= \int -4\sin\left(2t - \frac{\pi}{2}\right)\, dt \\
&= \int -2\sin u\, du \qquad \left(u = 2t - \tfrac{\pi}{2}, \quad du = 2\, dt\right) \\
&= 2\cos u + C_1 \\
&= 2\cos\left(2t - \frac{\pi}{2}\right) + C_1
\end{aligned}$$

We use the initial condition $s'(0) = 100$ to find the value of C_1.

$$100 = 2\cos\left(-\frac{\pi}{2}\right) + C_1 \qquad \Leftrightarrow \qquad 100 = C_1$$

Therefore we get:

$$s'(t) = 2\cos\left(2t - \frac{\pi}{2}\right) + 100$$

Now we do the whole process again to find $s(t)$.

$$\begin{aligned} s(t) &= \int \left[2\cos\left(2t - \frac{\pi}{2}\right) + 100\right]\, dt \\ &= \int [\cos u + 50]\, du && (u = 2t - \tfrac{\pi}{2}, \quad du = 2\, dt) \\ &= \sin u + 50u + C_2 \\ &= \sin\left(2t - \frac{\pi}{2}\right) + 50\left(2t - \frac{\pi}{2}\right) + C_2 \end{aligned}$$

We use the initial condition $s(0) = 0$ to find the value of C_2.

$$\begin{aligned} 0 = \sin\left(-\frac{\pi}{2}\right) + 50\left(-\frac{\pi}{2}\right) + C_2 \qquad &\Leftrightarrow \qquad 0 = -1 - 25\pi + C_2 \\ &\Leftrightarrow \qquad 1 + 25\pi = C_2 \end{aligned}$$

Therefore we get:

$$\begin{aligned} s(t) &= \sin\left(2t - \frac{\pi}{2}\right) + 50\left(2t - \frac{\pi}{2}\right) + 1 + 25\pi \\ &= \sin\left(2t - \frac{\pi}{2}\right) + 100t - 25\pi + 1 + 25\pi \\ &= \sin\left(2t - \frac{\pi}{2}\right) + 100t + 1 \end{aligned}$$