Section 6.5:

1. A force of 90N stretches a spring 1 m beyond its natural length. We can find the spring constant \( k \) by using Hooke’s Law:

\[ 90 = k \]

Thus we know that the force it takes to move the spring \( x \) meters beyond its natural length is given by \( F(x) = 90x \). To find the work it takes to stretch the spring 5 m beyond its natural length we take the following integral:

\[ W = \int_{0}^{5} F(x) \, dx \]
\[ = \int_{0}^{5} 90x \, dx \]
\[ = 45x^2 \bigg|_{0}^{5} \]
\[ = 1125 \text{ J} \]

2. Note that since the force is acting toward the origin, \( F(x) = -\frac{k}{x^2} \). To find the work done as the particle moves from point \( b \) to point \( a \), we can use the following integral:

\[ W = \int_{b}^{a} -\frac{k}{x^2} \, dx \]
\[ = k \left( \frac{1}{x} \right) \bigg|_{b}^{a} \]
\[ = \frac{k}{a} - \frac{k}{b} \]
\[ = k \frac{(b-a)}{ab} \]

3. a. Work to empty the tank by pumping the water back to ground level.

Consider a horizontal “slab” of water at level \( y \) with width \( \Delta y \). The force \( F_{\text{slab}} \) to lift the slab is given by:

\[ F_{\text{slab}} = 62.4 \cdot V_{\text{slab}} \]
\[ = 62.4 \cdot \Delta y \cdot 10 \cdot 12 \]
\[ = 62.4 \cdot 120 \Delta y \text{ lb} \]
To compute the work needed to pump this slab out of the tank, we recall that $F_{slab}$ must act over a distance of $y$ ft. Thus we have:

$$W_{slab} = F_{slab} \cdot d = 62.4 \cdot 120 y \Delta y \text{ ft} \cdot \text{lb}$$

To approximate the total work $W$ necessary to empty the tank, we could use a Riemann sum $f(y) = 7488y$ over the interval $0 \leq y \leq 20$.

$$W = \sum_{0}^{20} 64.2 \cdot 120 y \Delta y \text{ ft} \cdot \text{lb}$$

To find the exact value, we take the limit of the this sum over progressively finer partitions.

$$W = \int_{0}^{20} 64.2 \cdot 120 y \, dy$$

$$= \int_{0}^{20} 7488 y \, dy$$

$$= 3744 y^{2} \bigg|_{0}^{20}$$

$$= 3744 \cdot 400$$

$$= 1497600 \text{ ft} \cdot \text{lb}$$

b. The pump moves $250 \text{ ft} \cdot \text{lb/sec}$, the time it will take to empty the tank is:

$$\text{time} = \frac{1497600 \text{ ft} \cdot \text{lb}}{250 \text{ ft} \cdot \text{lb/sec}} = 5990.4 \text{ sec} \approx 1 \text{ hr 40 min}$$

c. The amount of work it takes to empty out the first half of the tank is given by:

$$\text{Work} = \int_{0}^{10} 7488 y \, dy$$

$$= 3744 y^{2} \bigg|_{0}^{10}$$

$$= 3744 \cdot 100$$

$$= 374400 \text{ ft-lb}$$

To see how much time this amount of work will take we use:

$$t = \frac{374400 \text{ ft-lb}}{250 \text{ ft} \cdot \text{lb/sec}} = 1497.6 \text{ sec}$$

The last thing to note is that $1497.6 \text{ sec} \approx 25 \text{ min}$. 

2
d. i. If water weighs 62.26 lb/ft$^3$

\[
W = \int_{0}^{20} 62.26 \cdot 120y \, dy = \int_{0}^{20} 7471.2y \, dy = 3735.2y^2|_{0}^{20} = 3735.6 \cdot 400 = 1494240 \text{ ft-lb}
\]

\[
t = \frac{1494240}{250} = 5976.96 \text{ sec} \approx 1 \text{ hr 40 min}
\]

ii. If water weighs 62.59 lb/ft$^3$

\[
W = \int_{0}^{20} 62.59 \cdot 120y \, dy = \int_{0}^{20} 7510.8y \, dy = 3755.4y^2|_{0}^{20} = 3755.4 \cdot 400 = 1502160 \text{ ft-lb}
\]

\[
t = \frac{1502160}{250} = 6008.64 \text{ sec} \approx 1 \text{ hr 40 min}
\]
4. a. Let $\rho$ be the $x$-coordinate of the second electron. Then $r^2 = (\rho - 1)^2$.

$$W = \int_{-1}^{0} F(r) \, dr$$
$$= \int_{-1}^{0} \frac{23 \times 10^{-29}}{(\rho - 1)^2} \, d\rho$$
$$= (23 \times 10^{-29}) \cdot \left[ \frac{1}{\rho - 1} \right]_{-1}^{0}$$
$$= \frac{1}{2} (23 \times 10^{-29})$$
$$= 11.5 \times 10^{-29}$$

b. We will use the fact that $W = W_1 + W_2$ where $W_1$ is the work against the fixed electron $(-1, 0)$ and $W_2$ is the work against the second fixed electron $(1, 0)$. We will let $\rho$ be the $x$-coordinate of the third electron. Then $r_1^2 = (\rho + 1)^2$ and $r_2^2 = (\rho - 1)^2$.

$$W_1 = \int_{3}^{5} \frac{23 \times 10^{-29}}{(\rho + 1)^2} \, d\rho$$
$$= -\frac{23 \times 10^{-29}}{\rho + 1} \bigg|_{3}^{5}$$
$$= -(23 \times 10^{-29}) \left( \frac{1}{5} - \frac{1}{4} \right)$$
$$= \frac{23}{12} \times 10^{-29}$$

$$W_2 = \int_{3}^{5} \frac{23 \times 10^{-29}}{(\rho - 1)^2} \, d\rho$$
$$= -\frac{23 \times 10^{-29}}{\rho - 1} \bigg|_{3}^{5}$$
$$= -(23 \times 10^{-29}) \left( \frac{1}{4} - \frac{1}{2} \right)$$
$$= \frac{23}{4} \times 10^{-29}$$

$$W = W_1 + W_2$$
$$= \frac{23}{12} \times 10^{-29} + \frac{23}{4} \times 10^{-29}$$
$$= \frac{23}{3} \times 10^{-29}$$