Math 21B - Homework Set 8

Section 8.1:

1. $\displaystyle\int t^2 \cos t \, dt$

   Let $u_1 = t^2, \quad du_1 = 2t\,dt$ and $v_1 = \sin t, \quad dv_1 = \cos t\,dt$

   Let $u_2 = 2t, \quad du_2 = 2dt$ and $v_2 = -\cos t, \quad dv_2 = \sin t\,dt$

$$\begin{aligned}\int t^2 \cos t\,dt &= u_1 v_1 - \int v_1\,du_1 \\ &= t^2 \sin t - \int 2t \sin t\,dt \\ &= t^2 \sin t - \left[u_2 v_2 - \int v_2\,du_2\right] \\ &= t^2 \sin t - \left[-2t\cos t + \int 2\cos t\,dt\right] \\ &= t^2 \sin t + 2t\cos t - 2\sin t + C\end{aligned}$$

2. $\displaystyle\int_1^e x^3 \ln x\,dx$

   Let $u = \ln x, \quad du = \frac{1}{x}\,dx$ and $v = \frac{1}{4}x^4, \quad dv = x^3\,dx$.

$$\begin{aligned}\int_1^e x^3 \ln x\,dx &= uv\big|_1^e - \int_1^e v\,du \\ &= \frac{1}{4}x^4 \ln x\bigg|_1^e - \int_1^e \frac{1}{4}x^4 \cdot \frac{1}{x}\,dx \\ &= \frac{1}{4}x^4 \ln x\bigg|_1^e - \frac{1}{4}\int_1^e x^3\,dx \\ &= \frac{1}{4}x^4 \ln x\bigg|_1^e - \frac{1}{16}x^4\bigg|_1^e \\ &= \left(\frac{e^4}{4} - 0\right) - \left(\frac{e^4}{16} - \frac{1}{16}\right) \\ &= \frac{3e^4 + 1}{16}\end{aligned}$$

3. $\displaystyle\int t^2 e^{4t}\,dt$

Let $u_1 = t^2$, $du_1 = 2t\,dt$ and $v_1 = \frac{1}{4}e^{4t}$, $dv_1 = e^{4t}\,dt$.

Let $u_2 = 2t$, $du_2 = 2dt$ and $v_2 = \frac{1}{16}e^{4t}$, $dv_2 = \frac{1}{4}e^{4t}\,dt$.

$$\begin{aligned}
\int t^2 e^{4t}\,dt &= u_1 v_1 - \int v_1\,du_1 \\
&= \frac{1}{4}t^2 e^{4t} - \int \frac{1}{4}2te^{4t}\,dt \\
&= \frac{1}{4}t^2 e^{4t} - \left[\frac{1}{8}te^{4t} - \int \frac{1}{8}e^{4t}\,dt\right] \\
&= \frac{1}{4}t^2 e^{4t} - \frac{1}{8}te^{4t} + \frac{1}{32}e^{4t} + C
\end{aligned}$$

4. $\int e^{\theta} \sin\theta\,d\theta$

Let $u_1 = \sin\theta$, $du_1 = \cos\theta\,d\theta$ and $v_1 = e^{\theta}$, $dv_1 = e^{\theta}\,d\theta$.

Let $u_2 = \cos\theta$, $du_2 = -\sin\theta\,d\theta$ and $v_2 = e^{\theta}$, $dv_2 = e^{\theta}\,d\theta$.

$$\begin{aligned}
\int e^{\theta}\sin\theta\,d\theta &= u_1 v_1 - \int v_1\,du_1 \\
&= e^{\theta}\sin\theta - \int e^{\theta}\cos\theta\,d\theta \\
&= e^{\theta}\sin\theta - \left[u_2 v_2 - \int v_2\,du_2\right] \\
&= e^{\theta}\sin\theta - \left[e^{\theta}\cos\theta + \int e^{\theta}\sin\theta\,d\theta\right] \\
&= e^{\theta}\sin\theta - e^{\theta}\cos\theta - \int e^{\theta}\sin\theta\,d\theta + C'
\end{aligned}$$

We can use this to solve for $\int e^{\theta}\sin\theta\,d\theta$.

$$\begin{aligned}
\int e^{\theta}\sin\theta\,d\theta &= e^{\theta}\sin\theta - e^{\theta}\cos\theta - \int e^{\theta}\sin\theta\,d\theta + C' \\
\Leftrightarrow 2\int e^{\theta}\sin\theta\,d\theta &= e^{\theta}\sin\theta - e^{\theta}\cos\theta + C' \\
\Leftrightarrow \int e^{\theta}\sin\theta\,d\theta &= \frac{e^{\theta}\sin\theta - e^{\theta}\cos\theta}{2} + C
\end{aligned}$$

5. $\displaystyle\int_0^{\pi/3} x\tan^2 x\,dx$

Let $u = x$, $\quad du = dx$ and $dv = \tan^2 x$, $\quad v = \int \tan^2 x\,dx = \int \sec^2 x - 1\,dx = \tan x - x$.

$$\begin{aligned}\int_0^{\pi/3} x\tan^2 x\,dx &= uv|_0^{\pi/3} - \int_0^{\pi/3} v\,du \\ &= \left(x\tan x - x^2\right)\Big|_0^{\pi/3} - \int_0^{\pi/3} \left(\tan x - x\right)\,dx \\ &= \left(x\tan x - x^2\right)\Big|_0^{\pi/3} - \left(\ln|\sec x| - \frac{1}{2}x^2\right)\Bigg|_0^{\pi/3} \\ &= \left(\frac{\pi\sqrt{3}}{3} - \frac{\pi^2}{9}\right) - \left(\ln(2) - \frac{\pi^2}{18}\right) \\ &= \frac{\pi\sqrt{3}}{3} - \frac{\pi^2}{18} - \ln(2)\end{aligned}$$

6. $\displaystyle\int z(\ln z)^2\,dz$

Let $w = \ln z$, $\quad dw = \frac{1}{z}\,dz$. Then we get that $z = e^w$.

$$\int z(\ln z)^2\,dz = \int \frac{z^2(\ln z)^2}{z}\,dz = \int w^2e^{2w}\,dw$$

Let $u_1 = w^2$, $\quad du_1 = 2w\,dw$ and $v_1 = \frac{1}{2}e^{2w}$, $\quad dv_1 = e^{2w}\,dw$.

Let $u_2 = w$, $\quad du_2 = dw$ and $v_2 = \frac{1}{2}e^{2w}$, $\quad dv_2 = e^{2w}\,dw$.

$$\begin{aligned}\int w^2e^{2w}\,dw &= u_1v_1 - \int v_1\,du_1 \\ &= \frac{1}{2}w^2e^{2w} - \int we^{2w}\,dw \\ &= \frac{1}{2}w^2e^{2w} - \left[u_2v_2 - \int v_2\,du_2\right] \\ &= \frac{1}{2}w^2e^{2w} - \left[\frac{1}{2}we^{2w} - \int \frac{1}{2}e^{2w}\,dw\right] \\ &= \frac{1}{2}w^2e^{2w} - \frac{1}{2}we^{2w} + \frac{1}{4}e^{2w} + C\end{aligned}$$

We now need to go back and substitute z back into the solution.

$$\int z(\ln z)^2\,dz = \frac{1}{2}z^2(\ln z)^2 - \frac{1}{2}z^2\ln z + \frac{1}{4}z^2 + C$$

7. Using integration by parts, with $u = \sin^{-1} x$ and $dv = dx$ (so that $v = x = \sin u$), we get

$$\begin{aligned}\int \sin^{-1} x\, dx &= x \sin^{-1} x - \int \sin(u)\, du \\ &= x \sin^{-1} x + \cos u + C \\ &= x \sin^{-1} x + \cos\left(\sin^{-1} x\right) + C \quad = x \sin^{-1} x + \sqrt{1 - x^2} + C\end{aligned}$$

Section 8.4:

1. $\displaystyle\int \frac{x+4}{x^2+5x-6}\, dx = \int \frac{x+4}{(x+6)(x-1)}\, dx$

$$\begin{aligned}\frac{x+4}{(x+6)(x-1)} = \frac{A}{x+6} + \frac{B}{x-1} \qquad &\Leftrightarrow \qquad x + 4 = (A+B)x + (-A+6B) \\ &\Leftrightarrow \qquad \begin{cases} 1 = A + B \\ 4 = -A + 6B \end{cases} \\ &\Leftrightarrow \qquad A = \frac{2}{7}, \quad B = \frac{5}{7}\end{aligned}$$

Therefore, going back to our integral we get:

$$\begin{aligned}\int \frac{x+4}{x^2+5x-6}\, dx &= \int \left(\frac{2}{7} \cdot \frac{1}{x+6} + \frac{5}{7} \cdot \frac{1}{x-1}\right) dx \\ &= \frac{2}{7} \ln|x+6| + \frac{5}{7} \ln|x-1| + C\end{aligned}$$

2. $\displaystyle\int \frac{x+3}{2x^3-8x}\, dx = \int \frac{x+3}{2x(x-2)(x+2)}\, dx$

$$\begin{aligned}\frac{x+3}{2x(x-2)(x+2)} = \frac{A}{2x} + \frac{B}{x-2} + \frac{C}{x+2} \qquad &\Leftrightarrow \qquad x + 3 = (A + 2B + 2C)x^2 + (4C - 4B)x - 4A \\ &\Leftrightarrow \qquad \begin{cases} 0 = A + 2B + 2C \\ 1 = 4C - 4B \\ 3 = -4A \end{cases} \\ &\Leftrightarrow \qquad A = -\frac{3}{4}, \quad B = \frac{1}{16}, \quad C = \frac{5}{16}\end{aligned}$$

Therefore, going back to our integral we get:

$$\begin{aligned}\int \frac{x+3}{2x^3-8x}\, dx &= \int \left(-\frac{3}{4} \cdot \frac{1}{2x} + \frac{1}{16} \cdot \frac{1}{x-2} + \frac{5}{16} \cdot \frac{1}{x+2}\right) dx \\ &= -\frac{3}{8} \ln|x| + \frac{1}{16} \ln|x-2| + \frac{5}{16} \ln|x+2| + C\end{aligned}$$

3. $\displaystyle\int_0^1 \frac{x^3}{x^2+2x+1}\,dx = \int_0^1 \left(x-2+\frac{3x+2}{x^2+2x+1}\right)dx \qquad \textit{(by long division)}$

$$\frac{3x+2}{x^2+2x+1} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \qquad \Leftrightarrow \qquad 3x+2 = Ax + (A+B)$$

$$\Leftrightarrow \qquad \begin{cases} 3 = A \\ 2 = A+B \end{cases}$$

$$\Leftrightarrow A = 3, \quad B = -1$$

Therefore, going back to our integral we get:

$$\begin{aligned}
\int_0^1 \frac{x^3}{x^2+2x+1}\,dx &= \int_0^1 \left(x-2+\frac{3}{x+1}-\frac{1}{(x+1)^2}\right)dx \\
&= \left(\frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1}\right)\Bigg|_0^1 \\
&= \left(\frac{1}{2} - 2 + 3\ln 2 + \frac{1}{2}\right) - (1) \\
&= 3\ln 2 - 2
\end{aligned}$$

4. $\displaystyle\int_0^1 \frac{dx}{(x+1)(x^2+1)}$

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \qquad \Leftrightarrow \qquad 1 = (A+B)x^2 + (B+C)x + (C+A)$$

$$\Leftrightarrow \qquad \begin{cases} 0 = A+B \\ 0 = B+C \\ 1 = C+A \end{cases}$$

$$\Leftrightarrow \qquad A = \frac{1}{2}, \quad B = -\frac{1}{2}, \quad C = \frac{1}{2}$$

Therefore, going back to our integral we get:

$$\begin{aligned}
\int_0^1 \frac{dx}{(x+1)(x^2+1)} &= \int_0^1 \left(\frac{1}{2}\cdot\frac{1}{x+1} + \frac{1}{2}\cdot\frac{1-x}{x^2+1}\right)dx \\
&= \frac{1}{2}\int_0^1 \left(\frac{1}{x+1} + \frac{1}{x^2+1} - \frac{x}{x^2+1}\right)dx \\
&= \frac{1}{2}\left(\ln|x+1| + \tan^{-1}x - \frac{1}{2}\ln|x^2+1|\right)\Bigg|_0^1 \\
&= \frac{1}{2}\left(\ln 2 + \frac{\pi}{4} - \frac{\ln 2}{2}\right) \\
&= \frac{1}{4}\ln 2 + \frac{\pi}{8}
\end{aligned}$$

5. $\displaystyle\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dt$

$$\frac{3t^2+t+4}{t(t^2+1)} = \frac{A}{t} + \frac{Bx+C}{t^2+1} \qquad \Leftrightarrow \qquad 3t^2+t+4 = (A+B)t^2 + Ct + A$$

$$\Leftrightarrow \qquad \begin{cases} 3 = A+B \\ 1 = C \\ 4 = A \end{cases}$$

$$\Leftrightarrow \qquad A = 4, \quad B = -1, \quad C = 1$$

Therefore, going back to our integral we get:

$$\begin{aligned}
\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dx &= \int_1^{\sqrt{3}} \left(\frac{4}{t} + \frac{1-t}{t^2+1}\right)\,dt \\
&= \int_1^{\sqrt{3}} \left(\frac{4}{t} + \frac{1}{t^2+1} - \frac{t}{t^2+1}\right)\,dt \\
&= \left(4\ln|t| + \tan^{-1} t - \frac{1}{2}\ln|t^2+1|\right)\Bigg|_1^{\sqrt{3}} \\
&= \left(4\ln(\sqrt{3}) + \frac{\pi}{3} - \frac{1}{2}\ln 4\right) - \left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right) \\
&= 2\ln 3 + \frac{\pi}{3} - \ln 2 - \frac{\pi}{4} + \frac{1}{2}\ln 2 \\
&= 2\ln 3 - \frac{1}{2}\ln 2 + \frac{\pi}{12}
\end{aligned}$$

6. $\displaystyle\int \frac{x^4}{x^2-1}\,dx = \int \left(x^2+1+\frac{1}{x^2-1}\right)\,dx$

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} \qquad \Leftrightarrow \qquad 1 = (A+B)x + (A-B)$$

$$\Leftrightarrow \qquad \begin{cases} 0 = A+B \\ 1 = A-B \end{cases}$$

$$\Leftarrow \qquad A = \frac{1}{2}, \quad B = -\frac{1}{2}$$

Therefore, going back to our integral we get:

$$\begin{aligned}
\int \frac{x^4}{x^2-1}\,dx &= \int \left(x^2+1+\frac{1}{2}\cdot\frac{1}{x-1} - \frac{1}{2}\cdot\frac{1}{x+1}\right)\,dx \\
&= \frac{1}{3}x^3 + x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C \\
&= \frac{1}{3}x^3 + x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C
\end{aligned}$$

Section 8.2:

1.

$$\begin{aligned}\int_0^{\pi/2} \sin^5 x\,dx &= \int_0^{\pi/2} \sin x\left(1-\cos^2 x\right)^2\,dx\\ &= \int_0^{\pi/2}\left(\sin x - 2\cos^2 x\sin x + \cos^4 x \sin x\right)\,dx\\ &= \left(-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x\right)\Bigg|_0^{\pi/2}\\ &= -\left(-1+\frac{2}{3}-\frac{1}{5}\right)\\ &= \frac{8}{15}\end{aligned}$$

2.

$$\begin{aligned}\int_0^1 8\cos^4(2\pi x)\,dx &= 8\int_0^1\left(\frac{\cos(4\pi x)+1}{2}\right)^2\,dx\\ &= 8\int_0^1 \frac{\cos^2(4\pi x)+2\cos(4\pi x)+1}{4}\,dx\\ &= 2\int_0^1\left(\frac{\cos(8\pi x)+1}{2}+2\cos(4\pi x)+1\right)\,dx\\ &= \int_0^1\left[\cos(8\pi x)+4\cos(4\pi x)+3\right]\,dx\\ &= \left(\frac{1}{8\pi}\sin(8\pi x)+\frac{1}{\pi}\sin(4\pi x)+3x\right)\Bigg|_0^1\\ &= 3\end{aligned}$$

3.

$$\begin{aligned}\int_0^{2\pi}\sqrt{\frac{1-\cos x}{2}}\,dx &= \int_0^{2\pi}\sqrt{\sin^2\left(\frac{x}{2}\right)}\,dx\\ &= \int_0^{2\pi}\left|\sin\left(\frac{x}{2}\right)\right|\,dx\\ &= \int_0^{2\pi}\sin\left(\frac{x}{2}\right)\,dx \qquad \left(\sin\left(\frac{x}{2}\right)\geq 0,\quad 0\leq x\leq 2\pi\right)\\ &= -2\cos\left(\frac{x}{2}\right)\Bigg|_0^{2\pi}\\ &= 2-(-2)\\ &= 4\end{aligned}$$

4.

$$\begin{aligned}
\int_{-\pi/4}^{\pi/4} \sqrt{\sec^2 x - 1}\, dx &= \int_{-\pi/4}^{\pi/4} \sqrt{\tan^2 x}\, dx \\
&= \int_{-\pi/4}^{\pi/4} |\tan x|\, dx \\
&= 2\int_{0}^{\pi/4} \tan x\, dx \\
&= \left(2\ln|\sec x|\right)\Big|_0^{\pi/4} \\
&= 2\ln(\sqrt{2}) \\
&= \ln 2
\end{aligned}$$

5.

$$\begin{aligned}
\int_{0}^{\pi/4} \sec^4 x\, dx &= \int_{0}^{\pi/4} \sec^2 x(\tan^2 x + 1)\, dx \\
&= \int_{0}^{\pi/4} (\sec^2 x \tan^2 x + \sec^2 x)\, dx \\
&= \left(\frac{1}{3}\tan^3 x + \tan x\right)\Bigg|_0^{\pi/4} \\
&= \frac{1}{3} + 1 \\
&= \frac{4}{3}
\end{aligned}$$

6.

$$\begin{aligned}
\int_{-\pi/4}^{\pi/4} 6\tan^4 x\, dx &= 6\int_{-\pi/4}^{\pi/4} \tan^2 x(\sec^2 - 1)\, dx \\
&= 6\int_{-\pi/4}^{\pi/4} (\tan^2 x \sec^2 x - \tan^2 x)\, dx \\
&= 6\int_{-\pi/4}^{\pi/4} (\tan^2 x \sec^2 x - \sec^2 x + 1)\, dx \\
&= 6\left(\frac{1}{3}\tan^3 x - \tan x + x\right)\Bigg|_{-\pi/4}^{\pi/4} \\
&= 6\left[\left(\frac{1}{3} - 1 + \frac{\pi}{4}\right) - \left(-\frac{1}{3} + 1 - \frac{\pi}{4}\right)\right] \\
&= 3\pi - 8
\end{aligned}$$

7.

$$\int_{-\pi}^{0} \sin(3x)\cos(2x)\,dx = \int_{-\pi}^{0} \left(\frac{1}{2}\sin x + \frac{1}{2}\sin(5x)\right)\,dx \qquad (\textit{see p465 })$$

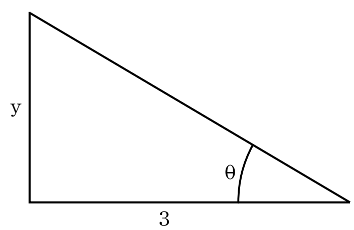
$$= \left(-\frac{1}{2}\cos x - \frac{1}{10}\cos(5x)\right)\Big|_{-\pi}^{0}$$

$$= \left(-\frac{1}{2} - \frac{1}{10}\right) - \left(\frac{1}{2} + \frac{1}{10}\right)$$

$$= -\frac{6}{5}$$

Section 8.3:

1. $\displaystyle\int \frac{1}{\sqrt{9+y^2}}\,dy$



Let $y = 3\tan\theta$, $\quad dy = 3\sec^2\theta\,d\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

$$\int \frac{1}{\sqrt{9+y^2}}\,dy = \int \frac{3\sec^2\theta}{\sqrt{9+9\tan^2\theta}}\,d\theta$$

$$= \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\sec\left(\tan^{-1}\left(\frac{y}{3}\right)\right) + \tan\left(\tan^{-1}\left(\frac{y}{3}\right)\right)\right| + C$$

$$= \ln\left|\frac{\sqrt{9+y^2}}{3} + \frac{y}{3}\right| + C \qquad \text{or} \qquad \ln\left|\sqrt{9+y^2} + y\right| + C'$$

2. $\displaystyle\int_0^2 \frac{dx}{8+2x^2}$

Let $x = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

$$\begin{aligned}
\int_0^2 \frac{dx}{8+2x^2} &= \int_{\tan^{-1}(0)}^{\tan^{-1}(1)} \frac{2\sec^2\theta}{8+8\tan^2\theta}\, d\theta \\
&= \int_0^{\pi/4} \frac{2\sec^2\theta}{8\sec^2\theta}\, d\theta \\
&= \int_0^{\pi/4} \frac{1}{4}\, d\theta \\
&= \left.\frac{\theta}{4}\right|_0^{\pi/4} \\
&= \frac{\pi}{16}
\end{aligned}$$

OR

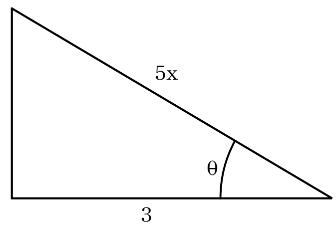
$$\begin{aligned}
\int_0^2 \frac{dx}{8+2x^2} &= \frac{1}{8}\int_0^2 \frac{dx}{1+\left(\frac{x}{2}\right)^2}\, dx \\
&= \left.\frac{1}{8}\cdot 2\tan^{-1}\left(\frac{x}{2}\right)\right|_0^2 \\
&= \frac{\pi}{16}
\end{aligned}$$

3. $\displaystyle\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}}$

Let $x = 3\sin\theta$, $dx = 3\cos\theta\, d\theta$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

$$\begin{aligned}
\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}} &= \int_0^{\pi/6} \frac{3\cos\theta}{\sqrt{9-9\sin^2\theta}}\, d\theta \\
&= \int_0^{\pi/6} \frac{3\cos\theta}{3\cos\theta}\, d\theta \\
&= \int_0^{\pi/6} d\theta \\
&= \theta|_0^{\pi/6} \\
&= \frac{\pi}{6}
\end{aligned}$$

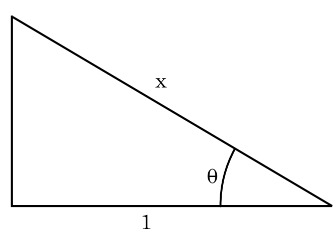
4. $\displaystyle\int \frac{5}{\sqrt{25x^2-9}}\, dx$ for $x > \frac{3}{5}$



Let $x = \frac{3}{5}\sec\theta$, $dx = \frac{3}{5}\sec\theta\tan\theta\, d\theta$ for $0 \le \theta < \frac{\pi}{2}$

$$\begin{aligned}
\int \frac{5}{\sqrt{25x^2-9}}\,dx &= \int \frac{5\cdot(3/5)\sec\theta\tan\theta}{\sqrt{25\cdot(9/25)\sec^2\theta-9}}\,d\theta \\
&= \int \frac{3\sec\theta\tan\theta}{3\sqrt{\sec^2\theta-1}}\,d\theta \\
&= \int \frac{\sec\theta\tan\theta}{\sqrt{\tan^2\theta}}\,d\theta \\
&= \int \sec\theta\,d\theta \\
&= \ln|\sec\theta+\tan\theta| + C \\
&= \ln\left|\sec\left(\sec^{-1}\left(\frac{5x}{3}\right)\right)+\tan\left(\sec^{-1}\left(\frac{5x}{3}\right)\right)\right| + C \\
&= \ln\left|\frac{5}{3}x+\frac{\sqrt{25x^2-9}}{3}\right| + C \qquad \text{or} \qquad \ln\left|5x+\sqrt{25x^2-9}\right| + C'
\end{aligned}$$

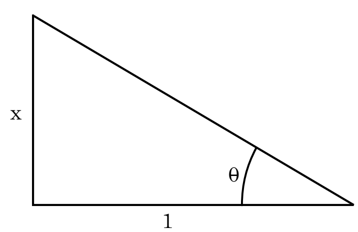
5. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-1}}\,dx$ for $x > 1$



Let $x = \sec\theta, \quad dx = \sec\theta\tan\theta\, d\theta$ for $0 \le \theta < \frac{\pi}{2}$.

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2-1}}\, dx &= \int \frac{\sec\theta\tan\theta}{\sec^2\theta\sqrt{\sec^2\theta - 1}}\, d\theta \\
&= \int \frac{\sec\theta\tan\theta}{\sec^2\theta|\tan\theta|}\, d\theta \\
&= \int \frac{\sec\theta\tan\theta}{\sec^2\theta\tan\theta}\, d\theta \qquad \left(\tan\theta \ge 0, \quad 0 \le \theta < \frac{\pi}{2}\right) \\
&= \int \cos\theta\, d\theta \\
&= \sin\theta + C \\
&= \sin\left(\sec^{-1}(x)\right) + C \\
&= \frac{\sqrt{x^2-1}}{x} + C
\end{aligned}$$

6. $\displaystyle\int \frac{1}{x^2\sqrt{x^2+1}}\, dx$



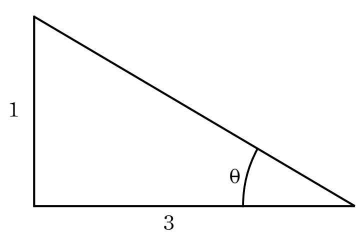
Let $x = \tan\theta$, $\quad dx = \sec^2\theta\, d\theta \quad$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

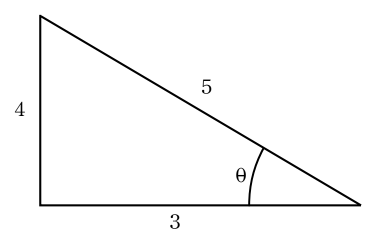
$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2+1}}\, dx &= \int \frac{\sec^2\theta}{\tan^2\theta\sqrt{\tan^2\theta+1}}\, d\theta \\
&= \int \frac{\sec^2\theta}{\tan^2\theta\sqrt{\sec^2\theta}}\, d\theta \\
&= \int \frac{\sec^2\theta}{\tan^2\theta\sec\theta}\, d\theta \qquad\qquad \left(\sec\theta > 1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right) \\
&= \int \frac{\sec\theta}{\tan^2\theta}\, d\theta \\
&= \int \frac{\cos\theta}{\sin^2\theta}\, d\theta \\
&= -\frac{1}{\sin\theta} + C \\
&= -\frac{1}{\sin\left(\tan^{-1}(x)\right)} + C \\
&= -\frac{\sqrt{x^2+1}}{x} + C
\end{aligned}$$

7. $\displaystyle\int_0^{\ln 4} \frac{e^t}{\sqrt{e^{2t}+9}}\, dt$

Firstly, let $x = e^t$, $\quad dx = e^t\, dt$.

$$\int_0^{\ln 4} \frac{e^t}{\sqrt{e^{2t}+9}}\, dt = \int_1^4 \frac{1}{\sqrt{x^2+9}}\, dx$$





Now, let $x = 3\tan\theta, \quad dx = 3\sec^2\theta\, d\theta$ for $\tan^{-1}\left(\frac{1}{3}\right) < \theta < \tan^{-1}\left(\frac{4}{3}\right)$.

$$\begin{aligned}
\int_1^4 \frac{1}{\sqrt{x^2+9}}\, dx &= \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \frac{3\sec^2\theta}{\sqrt{9\tan^2\theta+9}}\, d\theta \\
&= \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \frac{3\sec^2\theta}{3\sec\theta}\, d\theta \qquad \left(\sec\theta > 0, \quad \tan^{-1}\left(\frac{1}{3}\right) \le \theta \le \tan^{-1}\left(\frac{4}{3}\right)\right) \\
&= \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \sec\theta\, d\theta \\
&= \ln|\sec\theta + \tan\theta|\Big|_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \\
&= \ln\left|\sec\left(\tan^{-1}(4/3)\right) + \tan\left(\tan^{-1}(4/3)\right)\right| - \ln\left|\sec\left(\tan^{-1}(1/3)\right) + \tan\left(\tan^{-1}(1/3)\right)\right| \\
&= \ln\left(\frac{5}{3} + \frac{4}{3}\right) - \ln\left(\frac{\sqrt{10}}{3} + \frac{1}{3}\right) \\
&= \ln 3 - \ln\left(\sqrt{10}+1\right) + \ln 3 \\
&= 2\ln 3 - \ln\left(\sqrt{10}+1\right)
\end{aligned}$$

8. $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\, dx$

Let $x = \sec\theta, \quad dx = \sec\theta\tan\theta\, d\theta$ for $0 < \theta < \frac{\pi}{2}$

$$\begin{aligned}
\int \frac{1}{x\sqrt{x^2-1}}\, dx &= \int \frac{\sec\theta\tan\theta}{\sec\theta\sqrt{\sec^2\theta-1}}\, d\theta \\
&= \int \frac{\sec\theta\tan\theta}{\sec\theta\sqrt{\tan^2\theta}}\, d\theta \\
&= \int 1\, d\theta \\
&= \theta + C \\
&= \sec^{-1} x + C
\end{aligned}$$

9. $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\, dx$

Let $x = \sin\theta, \quad dx = \cos\theta \, d\theta \quad \text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\begin{aligned}
\int \frac{1}{\sqrt{1-x^2}}\,dx &= \int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta \\
&= \int \frac{\cos\theta}{\cos\theta}\,d\theta \qquad \left(\cos\theta > 0, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right) \\
&= \int 1\,d\theta \\
&= \theta + C \\
&= \sin^{-1} x + C
\end{aligned}$$

Section 8.7:

1.

$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt{x}}\,dx &= \lim_{b\to 0^+} \int_b^1 \frac{1}{\sqrt{x}}\,dx \\
&= \lim_{b\to 0^+} \; 2\sqrt{x}\Big|_b^1 \\
&= \lim_{b\to 0^+} \left(2 - 2\sqrt{b}\right) \\
&= 2
\end{aligned}$$

2.

$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx &= \lim_{b\to 1^-} \int_0^b \frac{1}{\sqrt{1-x^2}}\,dx \\
&= \lim_{b\to 1^-} \; \sin^{-1} x\Big|_0^b \\
&= \lim_{b\to 1^-} \left(\sin^{-1}(b) - \sin^{-1}(0)\right) \\
&= \frac{\pi}{2}
\end{aligned}$$

3.

$$\begin{aligned}
\int_1^\infty \frac{1}{x\sqrt{x^2-1}}\,dx &= \lim_{b\to 1^+} \int_b^2 \frac{1}{x\sqrt{x^2-1}}\,dx + \lim_{c\to\infty} \int_2^\infty \frac{1}{x\sqrt{x^2-1}}\,dx \\
&= \lim_{b\to 1^+} \; \sec^{-1} x\Big|_b^2 + \lim_{c\to\infty} \; \sec^{-1} x\Big|_2^c \\
&= \lim_{b\to 1^+} \left[\sec^{-1}(2) - \sec^{-1}(b)\right] + \lim_{c\to\infty} \left[\sec^{-1}(c) - \sec^{-1}(2)\right] \\
&= \frac{\pi}{3} - 0 + \frac{\pi}{2} - \frac{\pi}{3} \\
&= \frac{\pi}{2}
\end{aligned}$$

4. $\displaystyle\int_0^\infty 2e^{-\theta}\sin\theta\,d\theta$

We will first consider the indefinite integral $\displaystyle\int 2e^{-\theta}\sin\theta\,d\theta$.

Let $u_1 = \sin\theta,\quad du_1 = \cos\theta\,d\theta$ and $v_1 = -2e^{-\theta},\quad dv_1 = 2e^{-\theta}\,d\theta$.

Let $u_2 = \cos\theta,\quad du_2 = -\sin\theta\,d\theta$ and $v_2 = -2e^{-\theta},\quad dv_2 = 2e^{-\theta}\,d\theta$.

$$\begin{aligned}\int 2e^{-\theta}\sin\theta\,d\theta &= u_1v_1 - \int v_1\,du_1\\ &= -2e^{-\theta}\sin\theta + \int 2e^{-\theta}\sin\theta\,d\theta\\ &= -2e^{-\theta}\sin\theta + u_2v_2 - \int v_2\,du_2\\ &= -2e^{-\theta}\sin\theta - 2e^{-\theta}\cos\theta - \int 2e^{-\theta}\sin\theta\,d\theta\end{aligned}$$

Let $I = \displaystyle\int 2e^{-\theta}\sin\theta\,d\theta$. Solving for I we get:

$$\begin{aligned}I = -2e^{-\theta}\sin\theta - 2e^{-\theta}\cos\theta - I \qquad &\Leftrightarrow \qquad 2I = -2e^{-\theta}\sin\theta - 2e^{-\theta}\cos\theta\\ &\Leftrightarrow \qquad I = -e^{-\theta}\sin\theta - e^{-\theta}\cos\theta\end{aligned}$$

So we have that $\displaystyle\int 2e^{-\theta}\sin\theta\,d\theta = -e^{-\theta}\sin\theta - e^{-\theta}\cos\theta + C$. Now let's consider the improper integral $\displaystyle\int_0^\infty 2e^{-\theta}\sin\theta\,d\theta$.

$$\begin{aligned}\int_0^\infty 2e^{-\theta}\sin\theta\,d\theta &= \lim_{b\to\infty}\int_0^b 2e^{-\theta}\sin\theta\,d\theta\\ &= \lim_{b\to\infty}\left(-e^{-\theta}\sin\theta - e^{-\theta}\cos\theta\right)\Big|_0^b\\ &= \lim_{b\to\infty}[(-e^{-b}\sin b - e^{-b}\cos b) - (-1)]\\ &= 1\end{aligned}$$

5.

$$\begin{aligned}\int_{-\infty}^\infty 2xe^{-x^2}\,dx &= \lim_{b\to-\infty}\int_b^0 2xe^{-x^2}\,dx + \lim_{c\to\infty}\int_0^c 2xe^{-x^2}\,dx\\ &= \lim_{b\to-\infty} -e^{-x^2}\Big|_b^0 + \lim_{c\to\infty} -e^{-x^2}\Big|_0^c\\ &= \lim_{b\to-\infty}\left(-1+e^{-b^2}\right) + \lim_{c\to\infty}\left(-e^{-c^2}+1\right)\\ &= 0\end{aligned}$$

6.

$$
\begin{aligned}
\int_0^1 x\ln x\,dx &= \lim_{b\to 0^+}\int_b^1 x\ln x\,dx \\
&= \lim_{b\to 0^+}\left[\frac{1}{2}x^2\ln x\Big|_b^1 - \int_b^1 \frac{1}{2}x\,dx\right] && (\textit{Integration by Parts}) \\
&= \lim_{b\to 0^+}\left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right)\Big|_b^1 \\
&= \lim_{b\to 0^+}\left[\left(-\frac{1}{4}\right) - \left(\frac{1}{2}b^2\ln b - \frac{1}{4}b^2\right)\right] \\
&= -\frac{1}{4} - \frac{1}{2}\left[\lim_{b\to 0^+}\frac{\ln b}{\left(\frac{1}{b^2}\right)}\right] \\
&= -\frac{1}{4} - \frac{1}{2}\left[\lim_{b\to 0^+}\frac{\left(\frac{1}{b}\right)}{\left(\frac{-2}{b^3}\right)}\right] && (\textit{l'Hopital's Rule}) \\
&= -\frac{1}{4} + \frac{1}{4}\left[\lim_{b\to 0^+} b^2\right] \\
&= -\frac{1}{4}
\end{aligned}
$$

7.

$$
\begin{aligned}
\int_{-1}^4 \frac{1}{\sqrt{|x|}}\,dx &= \lim_{b\to 0^-}\int_{-1}^b \frac{1}{\sqrt{|x|}}\,dx + \lim_{c\to 0^+}\int_c^4 \frac{1}{\sqrt{|x|}}\,dx \\
&= \lim_{b\to 0^-}\int_{-1}^b \frac{1}{\sqrt{-x}}\,dx + \lim_{c\to 0^+}\int_c^4 \frac{1}{\sqrt{x}}\,dx \\
&= \lim_{b\to 0^-} -2\sqrt{-x}\Big|_{-1}^b + \lim_{c\to 0^+} 2\sqrt{x}\Big|_c^4 \\
&= \lim_{b\to 0^-}\left(-2\sqrt{-b}+2\right) + \lim_{c\to 0^+}\left(4-2\sqrt{c}\right) \\
&= 6
\end{aligned}
$$

Section 11.2:

1. $x = \cos t, y = 2 + \sin t, \quad 0 \leq t \leq 2\pi, \qquad x$-axis

$$\begin{aligned}
\text{Area} &= \int_0^{2\pi} 2\pi(2+\sin t)\sqrt{(-\sin t)^2 + (\cos t)^2}\, dt \\
&= 2\pi \int_0^{2\pi} (2+\sin t)\sqrt{\sin^2 t + \cos^2 t}\, dt \\
&= 2\pi \int_0^{2\pi} 2 + \sin t\, dt \\
&= \ 2\pi\, (2t - \cos t)|_0^{2\pi} \\
&= 2\pi[(4\pi - 1) - (0 - 1)] \\
&= 8\pi^2
\end{aligned}$$

2. $x = \ln(\sec t + \tan t) - \sin t, y = \cos t, \quad 0 \leq t \leq \frac{\pi}{3}; \qquad x$-axis

$$\begin{aligned}
\text{Area} &= \int_0^{\pi/3} 2\pi \cos t \sqrt{\left(\frac{\sec t \tan t + \sec^2 t}{\sec t + \tan t} - \cos t\right)^2 + (-\sin t)^2}\, dt \\
&= 2\pi \int_0^{\pi/3} \cos t \sqrt{\left(\frac{(\sec t)(\tan t + \sec t)}{\sec t + \tan t} - \cos t\right)^2 + \sin^2 t}\, dt \\
&= 2\pi \int_0^{\pi/3} \cos t \sqrt{(\sec t - \cos t)^2 + \sin^2 t}\, dt \\
&= 2\pi \int_0^{\pi/3} \cos t \sqrt{\sec^2 t - 2 + \cos^2 t + \sin^2 t}\, dt \\
&= 2\pi \int_0^{\pi/3} \cos t \sqrt{\sec^2 t - 1}\, dt \\
&= 2\pi \int_0^{\pi/3} \cos t \sqrt{\tan^2 t}\, dt \\
&= 2\pi \int_0^{\pi/3} \sin t\, dt \\
&= \ -2\pi \cos t|_0^{\pi/3} \\
&= -2\pi\left(\frac{1}{2} - 1\right) \\
&= \pi
\end{aligned}$$