Math 21B-B - Homework Set 2

Section 5.3:

1. (a) $\lim_{\|P\|\to 0} \sum_{k=1}^{n} ({c_k}^2 - 3c_k)\Delta x_k$, where P is a partition of $[-7, 5]$.

$$\int_{-7}^{5} \left(x^2 - 3x^2\right)\, dx$$

   (b) $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \sqrt{4 - {c_k}^2}\Delta x_k$, where P is a partition of $[0, 1]$.

$$\int_{0}^{1} \sqrt{4 - x^2}\, dx$$

   (c) $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (\tan c_k)\Delta x_k$, where P is a partition of $[0, \pi/4]$.

$$\int_{0}^{\pi/4} \tan(x)\, dx$$

2. Suppose that f and g are integrable and that:

$$\int_1^2 f(x)\, dx = -4, \qquad \int_1^5 f(x)\, dx = 6, \qquad \int_1^5 g(x) = 8.$$

   (a) $\int_2^2 g(x)\, dx = 0$

   Zero Width - Rule 2

   (b) $\int_5^1 g(x)\, dx = -\int_1^5 g(x)\, dx = -8$

   Order of Integration - Rule 1

   (c) $\int_1^2 3f(x)\, dx = 3\int_1^2 f(x)\, dx = 3 \cdot -4 = -12.$

   Constant Multiple - Rule 3

   (d) $\int_2^5 f(x)\, dx = \int_1^5 f(x)\, dx - \int_1^2 f(x)\, dx = 6 - (-4) = 10$

   Additivity - Rule 5

   (e) $\int_1^5 [f(x) - g(x)]\, dx = \int_1^5 f(x)\, dx - \int_1^5 g(x)\, dx = 6 - 8 = -2$

   Sum and Difference - Rule 4

(F) $\int_1^5 [4f(x) - g(x)]\,dx = \int_1^5 4f(x)\,dx - \int_1^5 g(x)\,dx = 4\int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx = 4 \cdot 6 - 8 = 24 - 8 = 16$

CONSTANT MULTIPLE AND SUM/DIFFERENCE - RULES 3 AND 4

3. Suppose that f and h are integrable and that:

$$\int_1^9 f(x)\,dx = -1, \qquad \int_7^9 f(x)\,dx = 5, \qquad \int_7^9 h(x)\,dx = 4.$$

(a) $\int_1^9 -2f(x)\,dx = -2\int_1^9 f(x)\,dx = -2 \cdot -1 = 2$

CONSTANT MULTIPLE - RULE 3

(B) $\int_7^9 [f(x) + h(x)]\,dx = \int_7^9 f(x)\,dx + \int_7^9 h(x)\,dx = 5 + 4 = 9$

SUM AND DIFFERENCE - RULE 4

(C) $\int_7^9 [2f(x) - 3h(x)]\,dx = \int_7^9 2f(x)\,dx - \int_7^9 3h(x)\,dx = 2\int_7^9 f(x)\,dx - 3\int_7^9 h(x)\,dx = 2 \cdot 5 - 3 \cdot 4 = 10 - 12 = -2$

CONSTANT MULTIPLE AND SUM/DIFFERENCE - RULES 3 AND 4

(D) $\int_9^1 f(x)\,dx = -\int_1^9 f(x)\,dx = -(-1) = 1$

ORDER OF INTEGRATION - RULE 1

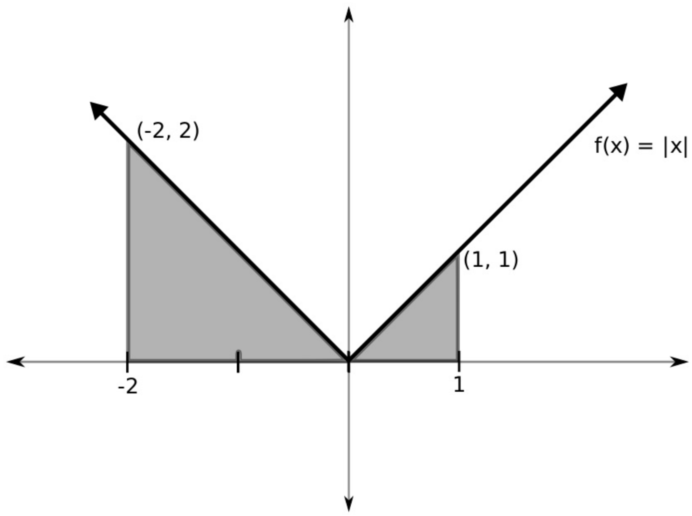
(E) $\int_1^7 f(x)\,dx = \int_1^9 f(x)\,dx - \int_7^9 f(x)\,dx = -1 - 5 = -6$

ADDITIVITY - RULE 5

(F) $\int_9^7 [h(x) - f(x)]\,dx = -\int_7^9 [h(x) - f(x)]\,dx = -\left[\int_7^9 h(x)\,dx - \int_7^9 f(x)\,dx\right] = -(4 - 5) = -(-1) = 1$

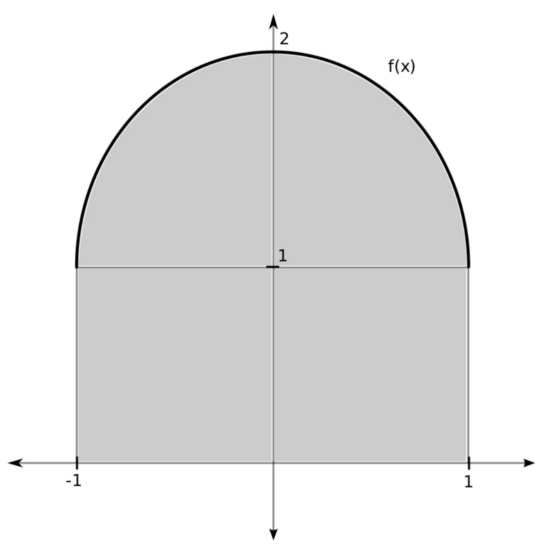
ORDER OF INTEGRATION AND SUM/DIFFERENCE - RULES 1 AND 4

4. $\int_{-2}^1 |x|\,dx$



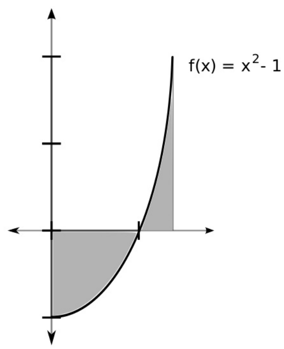
$$\begin{aligned}\int_{-2}^{1} |x|\,dx &= \text{Area of left triangle} + \text{Area of right triangle}\\ &= 1/2 \cdot 2 \cdot 2 + 1/2 \cdot 1 \cdot 1 = 2 + 1/2\\ &= 2 + 1/2\\ &= 5/2 \text{ square units}\end{aligned}$$

5. $\int_{-1}^{1} (1 + \sqrt{1 - x^2})\,dx$



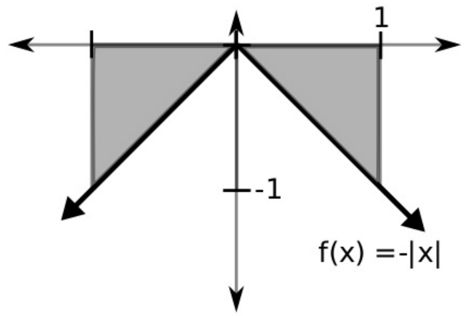
$$\int_{-1}^{1} (1+\sqrt{1-x^2})\,dx = \text{Area of semicircle} + \text{Area of Rectangle}$$
$$= 1/2 \cdot \pi \cdot 1^2 + 1 \cdot 2$$
$$= \frac{\pi}{2} + 2 \text{ square units}$$

6. (a) $f(x) = x^2 - 1$ on $[0, \sqrt{3}]$



$$av(f) = \left(\frac{1}{\sqrt{3}-0}\right) \int_0^{\sqrt{3}} (x^2-1)\,dx$$
$$= \left(\frac{1}{\sqrt{3}}\right) \left[\int_0^{\sqrt{3}} x^2\,dx - \int_0^{\sqrt{3}} 1\,dx\right]$$
$$= \left(\frac{1}{\sqrt{3}}\right) \left[\left(\frac{1}{3}\right) x^3 \Big|_0^{\sqrt{3}} - x \Big|_0^{\sqrt{3}}\right]$$
$$= \left(\frac{1}{\sqrt{3}}\right) \left[\left(\frac{1}{3}\right)(3\sqrt{3}-0) - (\sqrt{3}-0)\right]$$
$$= \left(\frac{1}{3}\right)(\sqrt{3}-\sqrt{3})$$
$$= 0$$

(b) $h(x) = -|x|$ on (i.) $[-1, 0]$, (ii.) $[0, 1]$, (iii.) $[-1, 1]$.



i.

$$\begin{aligned} av(f) &= \left(\frac{1}{0-(-1)}\right)\int_{-1}^{0} -|x|\,dx \\ &= (1)\int_{-1}^{0} -(-x)\,dx \\ &= \int_{-1}^{0} x\,dx \\ &= \left.\frac{1}{2}x^2\right|_{-1}^{0} \\ &= 0-\frac{1}{2} \\ &= -\frac{1}{2} \end{aligned}$$

ii.

$$\begin{aligned} av(f) &= \left(\frac{1}{1-0}\right)\int_{0}^{1} -|x|\,dx \\ &= (1)\int_{0}^{1} -x\,dx \\ &= \left.-\frac{1}{2}x^2\right|_{0}^{1} \\ &= -\frac{1}{2} \end{aligned}$$

iii.

$$\begin{aligned}
av(f) &= \left(\frac{1}{1-(-1)}\right)\int_{-1}^{1} -|x|\,dx \\
&= \left(\frac{1}{2}\right)\left[\int_{-1}^{0} -|x|\,dx + \int_{0}^{1} -|x|\,dx\right] \\
&= \left(\frac{1}{2}\right)\left[\int_{-1}^{0} -(-x)\,dx + \int_{0}^{1} -x\,dx\right] \\
&= \left(\frac{1}{2}\right)\left[\int_{-1}^{0} x\,dx + \int_{0}^{1} -x\,dx\right] \\
&= \left(\frac{1}{2}\right)\left[\frac{1}{2}x^2\bigg|_{-1}^{0} - \frac{1}{2}x^2\bigg|_{0}^{1}\right] \\
&= \left(\frac{1}{2}\right)\left[\left(0-\left(\frac{1}{2}\right)\right) - \left(\frac{1}{2}-0\right)\right] \\
&= \left(\frac{1}{2}\right)\left[-\frac{1}{2}-\frac{1}{2}\right] \\
&= \left(\frac{1}{2}\right)(-1) \\
&= -\frac{1}{2}
\end{aligned}$$

7. (a) Consider the partition P that subdivides the interval $[a,b]$ into n subintervals of width $\Delta x = \frac{b-a}{n}$ and let c_k be the right endpoint of each subinterval. So the partition is

$$P = \{a, a + \frac{b-a}{n}, a + \frac{2(b-a)}{n}, \ldots, a + \frac{n(b-a)}{n} = b\}$$

and $c_k = a + \frac{k(b-a)}{n}$. We get the Riemann sum

$$\begin{aligned}
\sum_{k=1}^{n} f(c_k)\Delta x &= \sum_{k=1}^{n} c_k^2 \left(\frac{b-a}{n}\right) \\
&= \frac{b-a}{n}\sum_{k=1}^{n}\left(a+\frac{k(b-a)}{n}\right) \\
&= \frac{b-a}{n}\sum_{k=1}^{n}\left(a^2+\frac{2ak(b-a)}{n}+\frac{k^2(b-a)^2}{n^2}\right) \\
&= \frac{b-a}{n}\left(\sum_{k=1}^{n} a^2 + \frac{2a(b-a)}{n}\sum_{k=1}^{n} k + \frac{(b-a)^2}{n^2}\sum_{k=1}^{n} k^2\right) \\
&= \frac{b-a}{n}\cdot na^2 + \frac{2a(b-a)^2}{n^2}\cdot\frac{n(n+1)}{2} + \frac{(b-a)^3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} \\
&= (b-a)a^2 + a(b-a)^2\cdot\frac{n+1}{n} + \frac{(b-a)^3(2n^2+3n+1)}{6n^2} \\
&= (b-a)a^2 + a(b-a)^2\cdot\left(1+\frac{1}{n}\right) + (b-a)^3\left(\frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}\right).
\end{aligned}$$

As $n \to \infty$ and $||P|| \to 0$, this expression converges to

$$(b-a)a^2 + a(b-a)^2 + (b-a)^3\cdot\frac{2}{6},$$

which simplifies to $\frac{b^3}{3} - \frac{a^3}{3}$. Thus $\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}$.

(b) Consider the partition P that subdivides the interval $[-1, 0]$ into n subintervals of width $\Delta x = \frac{1}{n}$ and let c_k be the right endpoint of each subinterval. So the partition is

$$P = \{-1, -1+\frac{1}{n}, -1+\frac{2}{n}, \ldots, -1+\frac{n}{n} = 0\}$$

and $c_k = -1 + \frac{k}{n}$. We get the Riemann sum

$$\begin{aligned}
\sum_{k=1}^{n} f(c_k)\Delta x &= \sum_{k=1}^{n}\Big((-1+\frac{k}{n}) - (-1+\frac{k}{n})^2\Big)\cdot\frac{1}{n} \\
&= \frac{1}{n}\sum_{k=1}^{n}(-1+\frac{k}{n} - 1 + \frac{2k}{n} - \left(\frac{k}{n}\right)^2) \\
&= \frac{-2}{n}\sum_{k=1}^{n} 1 + \frac{3}{n^2}\sum_{k=1}^{n} k - \frac{1}{n^3}\sum_{k=1}^{n} k^2 \\
&= \frac{-2}{n}n + \frac{3}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= -2 + \frac{3}{2}\left(1+\frac{1}{n}\right) + -\frac{1}{n^3}\left(\frac{2n^3+3n^2+n}{6n^3}\right).
\end{aligned}$$

As $n \to \infty$ and $||P|| \to 0$, this expression converges to $-2 + \frac{3}{2} - \frac{2}{6} = -\frac{5}{6}$. Thus $\int_{-1}^{0} (x - x^2)\, dx = -\frac{5}{6}$.

8. Use the Max-Min Inequality to find upper and lower bounds for the value of

$$\int_0^1 \frac{1}{1+x^2}\, dx$$

Notice that the integrand $f(x) = \frac{1}{1+x^2}$ is a decreasing function ($f'(x) = \frac{-2x}{(1+x^2)^2} \leq 0$ for $x \in [0,1]$). Therefore, on the interval f has a maximum at $x = 0$ and has a minimum at $x = 1$.

$$\max f = f(0) = 1$$
$$\min f = f(1) = \tfrac{1}{2}$$

Thus the Max-Min Inequality gives:

$$\tfrac{1}{2} \leq \int_0^1 \frac{1}{1+x^2}\, dx \leq 1$$

9. Use the Max-Min Inequality to find upper and lower bounds for the the value of

$$\int_0^{0.5} \frac{1}{1+x^2}\, dx \qquad \text{and} \qquad \int_{0.5}^1 \frac{1}{1+x^2}\, dx$$

Using the observation (from 65) that $f(x) = \frac{1}{1+x^2}$ is decreasing on $[0, 0.5]$ and $[0.5, 1]$, we get the following two inequalities:

$$0.4 \leq \int_0^{0.5} \frac{1}{1+x^2}\, dx \leq 0.5$$
$$0.25 \leq \int_{0.5}^1 \frac{1}{1+x^2}\, dx \leq 0.4.$$

Therefore, by adding the two inequalities we get the new (and improved) estimate:

$$0.65 \leq \int_0^1 \frac{1}{1+x^2}\, dx \leq 0.9.$$

10. We know that $\sin x \leq x$ for $x \geq 0$. Therefore we can get the upper bound:

$$\int_0^1 \sin x\,dx \le \int_0^1 x\,dx = \frac{1}{2}\cdot 1^2 - \frac{1}{2}\cdot 0^2 = \frac{1}{2}.$$

Section 5.4:

1. (a) $\displaystyle\int_{-2}^{0} (2x+5)\,dx = \left.(x^2+5x)\right|_{-2}^{0} = (0+0)-(4-10) = 6$

   (b) $\displaystyle\int_{-3}^{4} \left(5-\frac{x}{2}\right)\,dx = \left.\left(5x-\frac{x^2}{4}\right)\right|_{-3}^{4} = (20-4)-\left(-15-\frac{9}{4}\right) = 16 + \frac{69}{4} = \frac{133}{4}$

   (c) $\displaystyle\int_0^{\pi} \sin x\,dx = \left.-\cos x\right|_0^{\pi} = -\cos(\pi)+\cos(0) = 1+1 = 2$

   (d) $\displaystyle\int_0^{\pi/3} 2\sec^2 x\,dx = \left.2\tan x\,\right|_0^{\pi/3} = 2\tan(\pi/3) - 2\tan(0) = 2\cdot\sqrt{3} - 2\cdot 0 = 2\sqrt{3}$

   (e) $\displaystyle\int_1^{-1} (r+1)^2\,dr = -\int_{-1}^{1} (r^2+2r+1)\,dr = -\left[\left.\left(\frac{1}{3}r^3+r^2+r\right)\right|_{-1}^{1}\right] = -\left[\left(\frac{1}{3}+1+1\right)-\left(-\frac{1}{3}+1-1\right)\right] = -\left[\frac{7}{3}+\frac{1}{3}\right] = -\frac{8}{3}$

   (f) $\displaystyle\int_1^2 \left(\frac{1}{x}-e^{-x}\right)\,dx = \left.\left(\ln x + e^{-x}\right)\right|_1^2 = (\ln 2 + e^{-2}) - (0+e^{-1})$ $\ln 2 + \frac{1}{e^2} - \frac{1}{e}$

   (g) $\displaystyle\int_0^1 \frac{4}{1+x^2}\,dx = \left.4\arctan x\,\right|_0^1 = 4\arctan(1) - 4\arctan(0) = 4\cdot\frac{\pi}{4} - 4\cdot 0 = \pi$

   (h) $\displaystyle\int_2^5 \frac{x}{\sqrt{1+x^2}}\,dx = \int_2^5 x(1+x^2)^{-1/2}\,dx = \left.\frac{1}{2}\cdot 2\cdot\sqrt{1+x^2}\right|_2^5 = \left.\sqrt{1+x^2}\right|_2^5 = \sqrt{26}-\sqrt{5}$

(i) $\int_0^1 xe^{x^2}\,dx$

Notice that $\frac{d}{dx}x^2 = 2x$, and we have an x^2 in the exponent and an x in the product. Thus, keeping in mind the chain rule we get:

$f(x) = xe^{x^2} \qquad \Rightarrow \qquad F(x) = \frac{1}{2}e^{x^2}$

$$\int_0^1 xe^{x^2}\,dx = F(1) - F(0) = \frac{1}{2}e - \frac{1}{2} = \frac{1}{2}(e-1).$$

(j) $\int_1^2 \frac{\ln x}{x}\,dx$

Notice that $\frac{d}{dx}\ln x = \frac{1}{x}$, and we have an $\ln x$ and a $\frac{1}{x}$. Thus, keeping in mind the chain rule we get:

$f(x) = \frac{\ln x}{x} \qquad \Rightarrow \qquad F(x) = \frac{1}{2}\left(\ln x\right)^2$

$$\int_1^2 \frac{\ln x}{x} = F(2) - F(1) = \frac{1}{2}\left(\ln 2\right)^2 - 0 = \frac{1}{2}\left(\ln 2\right)^2.$$

2. (a) $\frac{d}{dx}\int_0^{\sqrt{x}} \cos t\,dt$

i.

$$\begin{aligned}\int_0^{\sqrt{x}} \cos t\,dt &= \sin t\,\big|_0^{\sqrt{x}} \\ &= \sin(\sqrt{x}) - \sin(0) \\ &= \sin(\sqrt{x})\end{aligned}$$

Therefore we get:

$$\begin{aligned}\frac{d}{dx}\int_0^{\sqrt{x}} \cos t\,dt &= \frac{d}{dx}\sin(\sqrt{x}) \\ &= \cos(\sqrt{x})\cdot\frac{1}{2\sqrt{x}}\end{aligned}$$

ii. Let $u = \sqrt{x}$

$$\begin{aligned}\frac{d}{dx}\int_0^{\sqrt{x}} \cos t\,dt &= \left(\frac{d}{du}\int_0^{u} \cos t\,dt\right)\cdot\frac{du}{dx} \\ &= \cos u\cdot\frac{du}{dx} \\ &= \cos(\sqrt{x})\cdot\frac{1}{2\sqrt{x}}\end{aligned}$$

(b) $\frac{d}{d\theta}\int_0^{\tan\theta} \sec^2 y\,dy$

i.

$$\begin{aligned}\int_0^{\tan\theta} \sec^2 y\,dy &= \tan y\,\big|_0^{\tan\theta} \\ &= \tan(\tan\theta) - \tan(0) \\ &= \tan(\tan\theta)\end{aligned}$$

Therefore we get:

$$\begin{aligned}\frac{d}{d\theta}\int_0^{\tan\theta} \sec^2 y\,dy &= \frac{d}{d\theta}\tan(\tan\theta) \\ &= \sec^2(\tan\theta)\cdot\sec^2\theta\end{aligned}$$

ii. Let $u = \tan\theta$

$$\begin{aligned}\frac{d}{d\theta}\int_0^{\tan\theta} \sec^2 y\,dy &= \left(\frac{d}{du}\int_0^{u} \sec^2 y\,dy\right)\cdot\frac{du}{d\theta} \\ &= \sec^2 u\cdot\frac{du}{d\theta} \\ &= \sec^2(\tan\theta)\cdot\sec^2\theta\end{aligned}$$

3. $y = \displaystyle\int_{\sqrt{x}}^{0} \sin\left(t^2\right)\,dt = -\int_0^{\sqrt{x}} \sin\left(t^2\right)\,dt$

Let $u = \sqrt{x}$. Then we get:

$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = -\sin(u^2)\cdot\frac{du}{dx} = -\sin(x)\cdot\frac{1}{x\sqrt{x}} = -\frac{\sin(x)}{2\sqrt{x}}$

4. (a) $y = -x^2 - 2x \quad$ where $\quad -3 \le x \le 2$

First, notice that $y = -x^2 - 2x = -x(x+2)$. Therefore in the interval $[-3, 2]$, the function has zeroes at $x = 0$ and $x = -2$.

$$\begin{aligned}\text{Area} &= -\int_{-3}^{-2}\left(-x^2-2x\right)dx + \int_{-2}^{0}\left(-x^2-2x\right)dx - \int_0^2\left(-x^2-2x\right)dx \\ &= -\left(-\frac{1}{3}x^3 - x^2\right)\Big|_{-3}^{-2} + \left(-\frac{1}{3}x^3 - x^2\right)\Big|_{-2}^{0} - \left(-\frac{1}{3}x^3 - x^2\right)\Big|_{0}^{2} \\ &= -\left[\left(\frac{8}{3}-4\right) - (9-9)\right] + \left[(0-0) - \left(\frac{8}{3}-4\right)\right] - \left[\left(-\frac{8}{3}-4\right) - (0-0)\right] \\ &= \frac{4}{3} + \frac{4}{3} + \frac{20}{3} \\ &= \frac{28}{3}\end{aligned}$$

(b) $y = 3x^2 - 3 \quad$ where $\quad -2 \leq x \leq 2$

First notice that $y = 3(x^2 - 1) = 3(x-1)(x+1)$. Therefore in the interval $[-2, 2]$, the function has zeroes at $x = \pm 1$.

$$
\begin{aligned}
\text{Area} &= \int_{-2}^{-1} \left(3x^2 - 3\right)\, dx - \int_{-1}^{1} \left(3x^2 - 2\right)\, dx + \int_{1}^{2} \left(3x^2 - 3\right)\, dx \\
&= \left(x^3 - 3x\right)\Big|_{-2}^{-1} - \left(x^3 - 3x\right)\Big|_{-1}^{1} + \left(x^3 - 3x\right)\Big|_{1}^{2} \\
&= [(-1+3) - (-8+6)] - [(1-3) - (-1+3)] + [(8-6) - (1-3)] \\
&= 4 + 4 + 4 \\
&= 12
\end{aligned}
$$

(c) $y = x^{1/3} - x \quad$ where $\quad -1 \leq x \leq 8$

First notice that $y = x^{1/3} - x = x^{1/3}\left(1 - x^{1/3}\right)\left(1 + x^{1/3}\right)$, so the on $[-1, 8]$ the function has zeroes at $x = 0, \pm 1$.

$$
\begin{aligned}
\text{Area} &= -\int_{-1}^{0} \left(x^{1/3} - x\right)\, dx + \int_{0}^{1} \left(x^{1/3} - x\right)\, dx - \int_{1}^{8} \left(x^{1/3} - x\right)\, dx \\
&= -\left(\frac{3}{4}x^{4/3} - \frac{1}{2}x^2\right)\Bigg|_{-1}^{0} + \left(\frac{3}{4}x^{4/3} - \frac{1}{2}x^2\right)\Bigg|_{0}^{1} - \left(\frac{3}{4}x^{4/3} - \frac{1}{2}x^2\right)\Bigg|_{1}^{8} \\
&= -\left[(0-0) - \left(\frac{3}{4} - \frac{1}{2}\right)\right] + \left[\left(\frac{3}{4} - \frac{1}{2}\right) - (0-0)\right] - \left[(12-32) - \left(\frac{3}{4} - \frac{1}{2}\right)\right] \\
&= \frac{1}{4} + \frac{1}{4} + \frac{81}{4} \\
&= \frac{83}{4}
\end{aligned}
$$

5. (a) $v = \frac{ds}{dt} = \frac{d}{dt}\int_{0}^{t} f(x)\, dx = f(t)$.

Therefore $v(5) = f(5) = 2$ m/s.

(b) Recall that the acceleration is the rate of change of the velocity, so $a = \frac{df}{dt}$. Since f has a negative slope at $t = 5$, the acceleration is negative at $t = 5$.

(c) Since $s = \int_{0}^{t} f(x)\, dx$, we know that the position at $t = 3$ can be represented as the area under the graph on $[0, 3]$. Notice that on this interval, f is a straight line and so the value of the integral is just the area of a triangle formed by $f(x)$, the x-axis and $x = 3$.

Therefore $s(3) = \frac{1}{2} \cdot 3 \cdot 3 = \frac{9}{2}$

(d) We want to know when s has a maximum and we will determine this by looking at $f(x)$, which is the derivative of the s. We know f is positive on $[0, 6]$ and negative on $[6, 9]$, which means that s goes from increasing to decreasing at $t = 6$. By the first derivative test, we know that s must have relative max at $t = 6$.

(e) At $x \approx 4$ and $x \approx 7$ since $s''(x) = f'(x)$ is zero at these points.

(f) The particle is moving towards the origin when it has negative velocity $\rightarrow f$ is negative. Hence it is moving towards the origin on $[6, 9]$. Similarly the particle is moving away from the origin when f is positive, hence on $[0, 6]$.

(g) Judging by the graph of f, we know that the particle is to the right of the origin (on the positive side) because the $s(9) = \displaystyle\int_0^9 f(x)\,dx > 0$ (the area above the x-axis is greater than the area below it).

6. Find $\lim_{x \to 0} \frac{1}{x^3} \displaystyle\int_0^x \frac{t^2}{t^4+1}\,dt$

Notice that $\lim_{x \to 0} \displaystyle\int_0^x \frac{t^2}{t^4+1}\,dt = 0$ by the rules of integrals and $\lim_{x \to 0} x^3 = 0$. Thus if you think of the expression inside the limit as a fraction, we have that both the numerator and the denominator go to 0. Hence we can apply l'Hôpital's Rule to the limit and get:

$$\begin{aligned}
\lim_{x \to 0} \frac{1}{x^3} \int_0^x \frac{t^2}{t^4+1}\,dt &= \lim_{x \to 0} \left[\left(\int_0^x \frac{t^2}{t^4+1}\,dt \right) / x^3 \right] \\
&= \lim_{x \to 0} \left[\left(\frac{x^2}{x^4+1} \right) / \left(3x^2\right) \right] \\
&= \lim_{x \to 0} \frac{1}{3\left(x^4+1\right)} \\
&= \frac{1}{3}
\end{aligned}$$

7. Suppose $f'(x) \geq 0$ for all values of x, and that $f(1) = 0$. Define:

$$g(x) = \int_0^x f(t)\,dt$$

(a) True. It follows from the FTC, part I.

(b) True. Differentiable $\Rightarrow$ Continuous.

(c) True. We know that $g'(1) = f(1) = 0$, so the tangent line at $x = 0$ is horizontal.

(d) False. We know that f crosses the x-axis at $x = 1$ and that f is always increasing. Therefore $g'(x) = f(x)$ is negative for $x < 1$ and $g'(x) = f(x)$ is positive for $x > 1$. By the first derivative test, this is a minimum, not a max.

Note that you could also use the fact that $g'(1) = 0$ and $g''(1) > 0$

(e) True. See (d.)

(f) False. $g''(x) = f'(x) > 0$, and hence does not change sign.

(g) True. $g'(x) = f(x)$ and we know that the $f(1) = 0$ and is increasing.