

Math 21B - Homework Set 8

Section 8.1:

1. $\int t^2 \cos t \, dt$

Let $u_1 = t^2$, $du_1 = 2t \, dt$ and $v_1 = \sin t$, $dv_1 = \cos t \, dt$

Let $u_2 = 2t$, $du_2 = 2 \, dt$ and $v_2 = -\cos t$, $dv_2 = \sin t \, dt$

$$\begin{aligned}\int t^2 \cos t \, dt &= u_1 v_1 - \int v_1 \, du_1 \\&= t^2 \sin t - \int 2t \sin t \, dt \\&= t^2 \sin t - \left[u_2 v_2 - \int v_2 \, du_2 \right] \\&= t^2 \sin t - \left[-2t \cos t + \int 2 \cos t \, dt \right] \\&= t^2 \sin t + 2t \cos t - 2 \sin t + C\end{aligned}$$

2. $\int_1^e x^3 \ln x \, dx$

Let $u = \ln x$, $du = \frac{1}{x} \, dx$ and $v = \frac{1}{4}x^4$, $dv = x^3 \, dx$.

$$\begin{aligned}\int_1^e x^3 \ln x \, dx &= uv \Big|_1^e - \int_1^e v \, du \\&= \frac{1}{4}x^4 \ln x \Big|_1^e - \int_1^e \frac{1}{4}x^4 \cdot \frac{1}{x} \, dx \\&= \frac{1}{4}x^4 \ln x \Big|_1^e - \frac{1}{4} \int_1^e x^3 \, dx \\&= \frac{1}{4}x^4 \ln x \Big|_1^e - \frac{1}{16}x^4 \Big|_1^e \\&= \left(\frac{e^4}{4} - 0 \right) - \left(\frac{e^4}{16} - \frac{1}{16} \right) \\&= \frac{3e^4 + 1}{16}\end{aligned}$$

3. $\int t^2 e^{4t} \, dt$

Let $u_1 = t^2$, $du_1 = 2t dt$ and $v_1 = \frac{1}{4}e^{4t}$, $dv_1 = e^{4t} dt$.

Let $u_2 = 2t$, $du_2 = 2dt$ and $v_2 = \frac{1}{16}e^{4t}$, $dv_2 = \frac{1}{4}e^{4t} dt$.

$$\begin{aligned}\int t^2 e^{4t} dt &= u_1 v_1 - \int v_1 du_1 \\ &= \frac{1}{4} t^2 e^{4t} - \int \frac{1}{4} 2t e^{4t} dt \\ &= \frac{1}{4} t^2 e^{4t} - \left[\frac{1}{8} t e^{4t} - \int \frac{1}{8} e^{4t} dt \right] \\ &= \frac{1}{4} t^2 e^{4t} - \frac{1}{8} t e^{4t} + \frac{1}{32} e^{4t} + C\end{aligned}$$

4. $\int e^\theta \sin \theta d\theta$

Let $u_1 = \sin \theta$, $du_1 = \cos \theta d\theta$ and $v_1 = e^\theta$, $dv_1 = e^\theta d\theta$.

Let $u_2 = \cos \theta$, $du_2 = -\sin \theta d\theta$ and $v_2 = e^\theta$, $dv_2 = e^\theta d\theta$.

$$\begin{aligned}\int e^\theta \sin \theta d\theta &= u_1 v_1 - \int v_1 du_1 \\ &= e^\theta \sin \theta - \int e^\theta \cos \theta d\theta \\ &= e^\theta \sin \theta - \left[u_2 v_2 - \int v_2 du_2 \right] \\ &= e^\theta \sin \theta - \left[e^\theta \cos \theta + \int e^\theta \sin \theta d\theta \right] \\ &= e^\theta \sin \theta - e^\theta \cos \theta - \int e^\theta \sin \theta d\theta + C'\end{aligned}$$

We can use this to solve for $\int e^\theta \sin \theta d\theta$.

$$\begin{aligned}\int e^\theta \sin \theta d\theta &= e^\theta \sin \theta - e^\theta \cos \theta - \int e^\theta \sin \theta d\theta + C' \\ \Leftrightarrow 2 \int e^\theta \sin \theta d\theta &= e^\theta \sin \theta - e^\theta \cos \theta + C' \\ \Leftrightarrow \int e^\theta \sin \theta d\theta &= \frac{e^\theta \sin \theta - e^\theta \cos \theta}{2} + C\end{aligned}$$

5. $\int_0^{\pi/3} x \tan^2 x \, dx$

Let $u = x$, $du = dx$ and $dv = \tan^2 x$, $v = \int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx = \tan x - x$.

$$\begin{aligned} \int_0^{\pi/3} x \tan^2 x \, dx &= uv \Big|_0^{\pi/3} - \int_0^{\pi/3} v \, du \\ &= (x \tan x - x^2) \Big|_0^{\pi/3} - \int_0^{\pi/3} (\tan x - x) \, dx \\ &= (x \tan x - x^2) \Big|_0^{\pi/3} - \left(\ln |\sec x| - \frac{1}{2} x^2 \right) \Big|_0^{\pi/3} \\ &= \left(\frac{\pi\sqrt{3}}{3} - \frac{\pi^2}{9} \right) - \left(\ln(2) - \frac{\pi^2}{18} \right) \\ &= \frac{\pi\sqrt{3}}{3} - \frac{\pi^2}{18} - \ln(2) \end{aligned}$$

6. $\int z(\ln z)^2 \, dz$

Let $w = \ln z$, $dw = \frac{1}{z} dz$. Then we get that $z = e^w$.

$$\int z(\ln z)^2 \, dz = \int \frac{z^2(\ln z)^2}{z} \, dz = \int w^2 e^{2w} \, dw$$

Let $u_1 = w^2$, $du_1 = 2w \, dw$ and $v_1 = \frac{1}{2}e^{2w}$, $dv_1 = e^{2w} \, dw$.

Let $u_2 = w$, $du_2 = dw$ and $v_2 = \frac{1}{2}e^{2w}$, $dv_2 = e^{2w} \, dw$.

$$\begin{aligned} \int w^2 e^{2w} \, dw &= u_1 v_1 - \int v_1 \, du_1 \\ &= \frac{1}{2} w^2 e^{2w} - \int w e^{2w} \, dw \\ &= \frac{1}{2} w^2 e^{2w} - \left[u_2 v_2 - \int v_2 \, du_2 \right] \\ &= \frac{1}{2} w^2 e^{2w} - \left[\frac{1}{2} w e^{2w} - \int \frac{1}{2} e^{2w} \, dw \right] \\ &= \frac{1}{2} w^2 e^{2w} - \frac{1}{2} w e^{2w} + \frac{1}{4} e^{2w} + C \end{aligned}$$

We now need to go back and substitute z back into the solution.

$$\int z(\ln z)^2 \, dz = \frac{1}{2} z^2 (\ln z)^2 - \frac{1}{2} z^2 \ln z + \frac{1}{4} z^2 + C$$

7. Using integration by parts, with $u = \sin^{-1} x$ and $dv = dx$ (so that $v = x = \sin u$), we get

$$\begin{aligned}\int \sin^{-1} x \, dx &= x \sin^{-1} x - \int \sin(u) \, du \\ &= x \sin^{-1} x + \cos u + C \\ &= x \sin^{-1} x + \cos(\sin^{-1} x) + C = x \sin^{-1} x + \sqrt{1 - x^2} + C\end{aligned}$$

Section 8.4:

$$\begin{aligned}1. \int \frac{x+4}{x^2+5x-6} \, dx &= \int \frac{x+4}{(x+6)(x-1)} \, dx \\ \frac{x+4}{(x+6)(x-1)} &= \frac{A}{x+6} + \frac{B}{x-1} \quad \Leftrightarrow \quad x+4 = (A+B)x + (-A+6B) \\ &\Leftrightarrow \quad \begin{cases} 1 = A+B \\ 4 = -A+6B \end{cases} \\ &\Leftrightarrow \quad A = \frac{2}{7}, \quad B = \frac{5}{7}\end{aligned}$$

Therefore, going back to our integral we get:

$$\begin{aligned}\int \frac{x+4}{x^2+5x-6} \, dx &= \int \left(\frac{2}{7} \cdot \frac{1}{x+6} + \frac{5}{7} \cdot \frac{1}{x-1} \right) \, dx \\ &= \frac{2}{7} \ln|x+6| + \frac{5}{7} \ln|x-1| + C\end{aligned}$$

$$\begin{aligned}2. \int \frac{x+3}{2x^3-8x} \, dx &= \int \frac{x+3}{2x(x-2)(x+2)} \, dx \\ \frac{x+3}{2x(x-2)(x+2)} &= \frac{A}{2x} + \frac{B}{x-2} + \frac{C}{x+2} \quad \Leftrightarrow \quad x+3 = (A+2B+2C)x^2 + (4C-4B)x - 4A \\ &\Leftrightarrow \quad \begin{cases} 0 = A+2B+2C \\ 1 = 4C-4B \\ 3 = -4A \end{cases} \\ &\Leftrightarrow \quad A = -\frac{3}{4}, \quad B = \frac{1}{16}, \quad C = \frac{5}{16}\end{aligned}$$

Therefore, going back to our integral we get:

$$\begin{aligned}\int \frac{x+3}{2x^3-8x} \, dx &= \int \left(-\frac{3}{4} \cdot \frac{1}{2x} + \frac{1}{16} \cdot \frac{1}{x-2} + \frac{5}{16} \cdot \frac{1}{x+2} \right) \, dx \\ &= -\frac{3}{8} \ln|x| + \frac{1}{16} \ln|x-2| + \frac{5}{16} \ln|x+2| + C\end{aligned}$$

$$\begin{aligned}
3. \int_0^1 \frac{x^3}{x^2+2x+1} dx &= \int_0^1 \left(x-2 + \frac{3x+2}{x^2+2x+1} \right) dx \quad (\text{by long division}) \\
\frac{3x+2}{x^2+2x+1} &= \frac{A}{x+1} + \frac{B}{(x+1)^2} \quad \Leftrightarrow \quad 3x+2 = Ax + (A+B) \\
&\Leftrightarrow \quad \begin{cases} 3 = A \\ 2 = A+B \end{cases} \\
&\Leftrightarrow A = 3, \quad B = -1
\end{aligned}$$

Therefore, going back to our integral we get:

$$\begin{aligned}
\int_0^1 \frac{x^3}{x^2+2x+1} dx &= \int_0^1 \left(x-2 + \frac{3}{x+1} - \frac{1}{(x+1)^2} \right) dx \\
&= \left(\frac{x^2}{2} - 2x + 3 \ln|x+1| + \frac{1}{x+1} \right) \Big|_0^1 \\
&= \left(\frac{1}{2} - 2 + 3 \ln 2 + \frac{1}{2} \right) - (1) \\
&= 3 \ln 2 - 2
\end{aligned}$$

$$\begin{aligned}
4. \int_0^1 \frac{dx}{(x+1)(x^2+1)} \\
\frac{1}{(x+1)(x^2+1)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \quad \Leftrightarrow \quad 1 = (A+B)x^2 + (B+C)x + (C+A) \\
&\Leftrightarrow \quad \begin{cases} 0 = A+B \\ 0 = B+C \\ 1 = C+A \end{cases} \\
&\Leftrightarrow \quad A = \frac{1}{2}, \quad B = -\frac{1}{2}, \quad C = \frac{1}{2}
\end{aligned}$$

Therefore, going back to our integral we get:

$$\begin{aligned}
\int_0^1 \frac{dx}{(x+1)(x^2+1)} &= \int_0^1 \left(\frac{1}{2} \cdot \frac{1}{x+1} + \frac{1}{2} \cdot \frac{1-x}{x^2+1} \right) dx \\
&= \frac{1}{2} \int_0^1 \left(\frac{1}{x+1} + \frac{1}{x^2+1} - \frac{x}{x^2+1} \right) dx \\
&= \frac{1}{2} \left(\ln|x+1| + \tan^{-1} x - \frac{1}{2} \ln|x^2+1| \right) \Big|_0^1 \\
&= \frac{1}{2} \left(\ln 2 + \frac{\pi}{4} - \frac{\ln 2}{2} \right) \\
&= \frac{1}{4} \ln 2 + \frac{\pi}{8}
\end{aligned}$$

$$\begin{aligned}
5. \quad & \int_1^{\sqrt{3}} \frac{3t^2 + t + 4}{t^3 + t} dt \\
& \frac{3t^2 + t + 4}{t(t^2 + 1)} = \frac{A}{t} + \frac{Bx + C}{t^2 + 1} \quad \Leftrightarrow \quad 3t^2 + t + 4 = (A + B)t^2 + Ct + A \\
& \Leftrightarrow \quad \begin{cases} 3 = A + B \\ 1 = C \\ 4 = A \end{cases} \\
& \Leftrightarrow \quad A = 4, \quad B = -1, \quad C = 1
\end{aligned}$$

Therefore, going back to our integral we get:

$$\begin{aligned}
\int_1^{\sqrt{3}} \frac{3t^2 + t + 4}{t^3 + t} dx &= \int_1^{\sqrt{3}} \left(\frac{4}{t} + \frac{1-t}{t^2 + 1} \right) dt \\
&= \int_1^{\sqrt{3}} \left(\frac{4}{t} + \frac{1}{t^2 + 1} - \frac{t}{t^2 + 1} \right) dt \\
&= \left(4 \ln |t| + \tan^{-1} t - \frac{1}{2} \ln |t^2 + 1| \right) \Big|_1^{\sqrt{3}} \\
&= \left(4 \ln(\sqrt{3}) + \frac{\pi}{3} - \frac{1}{2} \ln 4 \right) - \left(\frac{\pi}{4} - \frac{1}{2} \ln 2 \right) \\
&= 2 \ln 3 + \frac{\pi}{3} - \ln 2 - \frac{\pi}{4} + \frac{1}{2} \ln 2 \\
&= 2 \ln 3 - \frac{1}{2} \ln 2 + \frac{\pi}{12}
\end{aligned}$$

$$\begin{aligned}
6. \quad & \int \frac{x^4}{x^2 - 1} dx = \int \left(x^2 + 1 + \frac{1}{x^2 - 1} \right) dx \\
& \frac{1}{x^2 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1} \quad \Leftrightarrow \quad 1 = (A + B)x + (A - B) \\
& \Leftrightarrow \quad \begin{cases} 0 = A + B \\ 1 = A - B \end{cases} \\
& \Leftrightarrow \quad A = \frac{1}{2}, \quad B = -\frac{1}{2}
\end{aligned}$$

Therefore, going back to our integral we get:

$$\begin{aligned}
\int \frac{x^4}{x^2 - 1} dx &= \int \left(x^2 + 1 + \frac{1}{2} \cdot \frac{1}{x - 1} - \frac{1}{2} \cdot \frac{1}{x + 1} \right) dx \\
&= \frac{1}{3} x^3 + x + \frac{1}{2} \ln |x - 1| - \frac{1}{2} \ln |x + 1| + C \\
&= \frac{1}{3} x^3 + x + \frac{1}{2} \ln \left| \frac{x - 1}{x + 1} \right| + C
\end{aligned}$$

Section 8.2:

1.

$$\begin{aligned}
 \int_0^{\pi/2} \sin^5 x \, dx &= \int_0^{\pi/2} \sin x (1 - \cos^2 x)^2 \, dx \\
 &= \int_0^{\pi/2} (\sin x - 2 \cos^2 x \sin x + \cos^4 x \sin x) \, dx \\
 &= \left(-\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x \right) \Big|_0^{\pi/2} \\
 &= - \left(-1 + \frac{2}{3} - \frac{1}{5} \right) \\
 &= \frac{8}{15}
 \end{aligned}$$

2.

$$\begin{aligned}
 \int_0^1 8 \cos^4(2\pi x) \, dx &= 8 \int_0^1 \left(\frac{\cos(4\pi x) + 1}{2} \right)^2 \, dx \\
 &= 8 \int_0^1 \frac{\cos^2(4\pi x) + 2 \cos(4\pi x) + 1}{4} \, dx \\
 &= 2 \int_0^1 \left(\frac{\cos(8\pi x) + 1}{2} + 2 \cos(4\pi x) + 1 \right) \, dx \\
 &= \int_0^1 [\cos(8\pi x) + 4 \cos(4\pi x) + 3] \, dx \\
 &= \left(\frac{1}{8\pi} \sin(8\pi x) + \frac{1}{\pi} \sin(4\pi x) + 3x \right) \Big|_0^1 \\
 &= 3
 \end{aligned}$$

3.

$$\begin{aligned}
 \int_0^{2\pi} \sqrt{\frac{1 - \cos x}{2}} \, dx &= \int_0^{2\pi} \sqrt{\sin^2 \left(\frac{x}{2} \right)} \, dx \\
 &= \int_0^{2\pi} \left| \sin \left(\frac{x}{2} \right) \right| \, dx \\
 &= \int_0^{2\pi} \sin \left(\frac{x}{2} \right) \, dx \quad \left(\sin \left(\frac{x}{2} \right) \geq 0, \quad 0 \leq x \leq 2\pi \right) \\
 &= -2 \cos \left(\frac{x}{2} \right) \Big|_0^{2\pi} \\
 &= 2 - (-2) \\
 &= 4
 \end{aligned}$$

4.

$$\begin{aligned}
 \int_{-\pi/4}^{\pi/4} \sqrt{\sec^2 x - 1} \, dx &= \int_{-\pi/4}^{\pi/4} \sqrt{\tan^2 x} \, dx \\
 &= \int_{-\pi/4}^{\pi/4} |\tan x| \, dx \\
 &= 2 \int_0^{\pi/4} \tan x \, dx \\
 &= (2 \ln |\sec x|) \Big|_0^{\pi/4} \\
 &= 2 \ln(\sqrt{2}) \\
 &= \ln 2
 \end{aligned}$$

5.

$$\begin{aligned}
 \int_0^{\pi/4} \sec^4 x \, dx &= \int_0^{\pi/4} \sec^2 x (\tan^2 x + 1) \, dx \\
 &= \int_0^{\pi/4} (\sec^2 x \tan^2 x + \sec^2 x) \, dx \\
 &= \left(\frac{1}{3} \tan^3 x + \tan x \right) \Big|_0^{\pi/4} \\
 &= \frac{1}{3} + 1 \\
 &= \frac{4}{3}
 \end{aligned}$$

6.

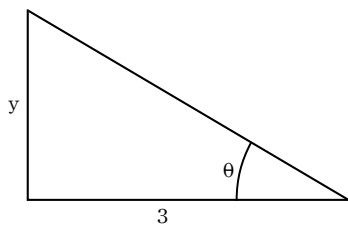
$$\begin{aligned}
 \int_{-\pi/4}^{\pi/4} 6 \tan^4 x \, dx &= 6 \int_{-\pi/4}^{\pi/4} \tan^2 x (\sec^2 x - 1) \, dx \\
 &= 6 \int_{-\pi/4}^{\pi/4} (\tan^2 x \sec^2 x - \tan^2 x) \, dx \\
 &= 6 \int_{-\pi/4}^{\pi/4} (\tan^2 x \sec^2 x - \sec^2 x + 1) \, dx \\
 &= 6 \left(\frac{1}{3} \tan^3 x - \tan x + x \right) \Big|_{-\pi/4}^{\pi/4} \\
 &= 6 \left[\left(\frac{1}{3} - 1 + \frac{\pi}{4} \right) - \left(-\frac{1}{3} + 1 - \frac{\pi}{4} \right) \right] \\
 &= 3\pi - 8
 \end{aligned}$$

7.

$$\begin{aligned}
 \int_{-\pi}^0 \sin(3x) \cos(2x) dx &= \int_{-\pi}^0 \left(\frac{1}{2} \sin x + \frac{1}{2} \sin(5x) \right) dx && (\text{see p465}) \\
 &= \left(-\frac{1}{2} \cos x - \frac{1}{10} \cos(5x) \right) \Big|_{-\pi}^0 \\
 &= \left(-\frac{1}{2} - \frac{1}{10} \right) - \left(\frac{1}{2} + \frac{1}{10} \right) \\
 &= -\frac{6}{5}
 \end{aligned}$$

Section 8.3:

1. $\int \frac{1}{\sqrt{9+y^2}} dy$



Let $y = 3 \tan \theta$, $dy = 3 \sec^2 \theta d\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

$$\begin{aligned}
 \int \frac{1}{\sqrt{9+y^2}} dy &= \int \frac{3 \sec^2 \theta}{\sqrt{9+9 \tan^2 \theta}} d\theta \\
 &= \int \sec \theta d\theta \\
 &= \ln |\sec \theta + \tan \theta| + C \\
 &= \ln \left| \sec \left(\tan^{-1} \left(\frac{y}{3} \right) \right) + \tan \left(\tan^{-1} \left(\frac{y}{3} \right) \right) \right| + C \\
 &= \ln \left| \frac{\sqrt{9+y^2}}{3} + \frac{y}{3} \right| + C \quad \text{or} \quad \ln \left| \sqrt{9+y^2} + y \right| + C'
 \end{aligned}$$

2. $\int_0^2 \frac{dx}{8+2x^2}$

Let $x = 2 \tan \theta$, $dx = 2 \sec^2 \theta d\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

$$\begin{aligned}
 \int_0^2 \frac{dx}{8+2x^2} &= \int_{\tan^{-1}(0)}^{\tan^{-1}(1)} \frac{2 \sec^2 \theta}{8+8 \tan^2 \theta} d\theta \\
 &= \int_0^{\pi/4} \frac{2 \sec^2 \theta}{8 \sec^2 \theta} d\theta \\
 &= \int_0^{\pi/4} \frac{1}{4} d\theta \\
 &= \left. \frac{\theta}{4} \right|_0^{\pi/4} \\
 &= \frac{\pi}{16}
 \end{aligned}$$

OR

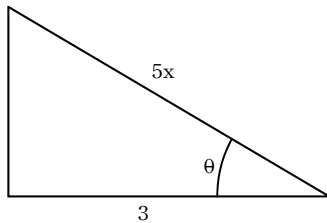
$$\begin{aligned}
 \int_0^2 \frac{dx}{8+2x^2} &= \frac{1}{8} \int_0^2 \frac{dx}{1+\left(\frac{x}{2}\right)^2} \\
 &= \frac{1}{8} \cdot 2 \tan^{-1} \left(\frac{x}{2} \right) \Big|_0^2 \\
 &= \frac{\pi}{16}
 \end{aligned}$$

3. $\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}}$

Let $x = 3 \sin \theta$, $dx = 3 \cos \theta d\theta$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

$$\begin{aligned}
 \int_0^{3/2} \frac{dx}{\sqrt{9-x^2}} &= \int_0^{\pi/6} \frac{3 \cos \theta}{\sqrt{9-9 \sin^2 \theta}} d\theta \\
 &= \int_0^{\pi/6} \frac{3 \cos \theta}{3 \cos \theta} d\theta \\
 &= \int_0^{\pi/6} d\theta \\
 &= \left. \theta \right|_0^{\pi/6} \\
 &= \frac{\pi}{6}
 \end{aligned}$$

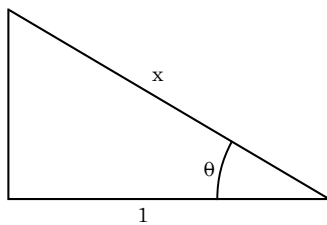
4. $\int \frac{5}{\sqrt{25x^2-9}} dx$ for $x > \frac{3}{5}$



Let $x = \frac{3}{5} \sec \theta$, $dx = \frac{3}{5} \sec \theta \tan \theta d\theta$ for $0 \leq \theta < \frac{\pi}{2}$

$$\begin{aligned}
 \int \frac{5}{\sqrt{25x^2 - 9}} dx &= \int \frac{5 \cdot (3/5) \sec \theta \tan \theta}{\sqrt{25 \cdot (9/25) \sec^2 \theta - 9}} d\theta \\
 &= \int \frac{3 \sec \theta \tan \theta}{3 \sqrt{\sec^2 \theta - 1}} d\theta \\
 &= \int \frac{\sec \theta \tan \theta}{\sqrt{\tan^2 \theta}} d\theta \\
 &= \int \sec \theta d\theta \\
 &= \ln |\sec \theta + \tan \theta| + C \\
 &= \ln \left| \sec \left(\sec^{-1} \left(\frac{5x}{3} \right) \right) + \tan \left(\sec^{-1} \left(\frac{5x}{3} \right) \right) \right| + C \\
 &= \ln \left| \frac{5}{3}x + \frac{\sqrt{25x^2 - 9}}{3} \right| + C \quad \text{or} \quad \ln |5x + \sqrt{25x^2 - 9}| + C'
 \end{aligned}$$

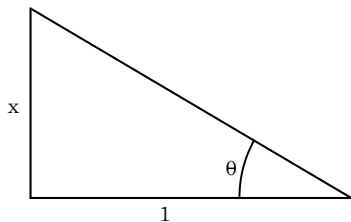
5. $\int \frac{1}{x^2 \sqrt{x^2 - 1}} dx$ for $x > 1$



Let $x = \sec \theta$, $dx = \sec \theta \tan \theta d\theta$ for $0 \leq \theta < \frac{\pi}{2}$.

$$\begin{aligned}
 \int \frac{1}{x^2 \sqrt{x^2 - 1}} dx &= \int \frac{\sec \theta \tan \theta}{\sec^2 \theta \sqrt{\sec^2 \theta - 1}} d\theta \\
 &= \int \frac{\sec \theta \tan \theta}{\sec^2 \theta |\tan \theta|} d\theta \\
 &= \int \frac{\sec \theta \tan \theta}{\sec^2 \theta \tan \theta} d\theta \quad \left(\tan \theta \geq 0, \quad 0 \leq \theta < \frac{\pi}{2} \right) \\
 &= \int \cos \theta d\theta \\
 &= \sin \theta + C \\
 &= \sin (\sec^{-1}(x)) + C \\
 &= \frac{\sqrt{x^2 - 1}}{x} + C
 \end{aligned}$$

6. $\int \frac{1}{x^2 \sqrt{x^2 + 1}} dx$



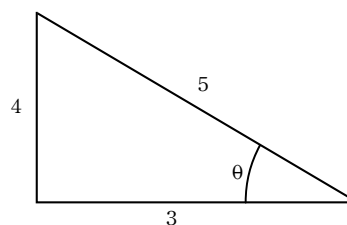
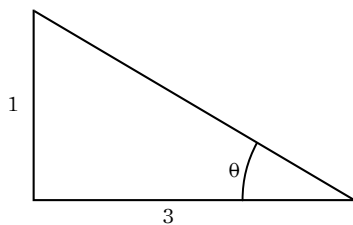
Let $x = \tan \theta$, $dx = \sec^2 \theta d\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\begin{aligned}
 \int \frac{1}{x^2 \sqrt{x^2 + 1}} dx &= \int \frac{\sec^2 \theta}{\tan^2 \theta \sqrt{\tan^2 \theta + 1}} d\theta \\
 &= \int \frac{\sec^2 \theta}{\tan^2 \theta \sqrt{\sec^2 \theta}} d\theta \\
 &= \int \frac{\sec^2 \theta}{\tan^2 \theta \sec \theta} d\theta \quad \left(\sec \theta > 1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \right) \\
 &= \int \frac{\sec \theta}{\tan^2 \theta} d\theta \\
 &= \int \frac{\cos \theta}{\sin^2 \theta} d\theta \\
 &= -\frac{1}{\sin \theta} + C \\
 &= -\frac{1}{\sin(\tan^{-1}(x))} + C \\
 &= -\frac{\sqrt{x^2 + 1}}{x} + C
 \end{aligned}$$

7. $\int_0^{\ln 4} \frac{e^t}{\sqrt{e^{2t} + 9}} dt$

Firstly, let $x = e^t$, $dx = e^t dt$.

$$\int_0^{\ln 4} \frac{e^t}{\sqrt{e^{2t} + 9}} dt = \int_1^4 \frac{1}{\sqrt{x^2 + 9}} dx$$



Now, let $x = 3 \tan \theta$, $dx = 3 \sec^2 \theta d\theta$ for $\tan^{-1} \left(\frac{1}{3} \right) < \theta < \tan^{-1} \left(\frac{4}{3} \right)$.

$$\begin{aligned}
\int_1^4 \frac{1}{\sqrt{x^2 + 9}} dx &= \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \frac{3 \sec^2 \theta}{\sqrt{9 \tan^2 \theta + 9}} d\theta \\
&= \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \frac{3 \sec^2 \theta}{3 \sec \theta} d\theta \quad \left(\sec \theta > 0, \quad \tan^{-1} \left(\frac{1}{3} \right) \leq \theta \leq \tan^{-1} \left(\frac{4}{3} \right) \right) \\
&= \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \sec \theta d\theta \\
&= \ln |\sec \theta + \tan \theta| \Big|_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \\
&= \ln |\sec(\tan^{-1}(4/3)) + \tan(\tan^{-1}(4/3))| - \ln |\sec(\tan^{-1}(1/3)) + \tan(\tan^{-1}(1/3))| \\
&= \ln \left(\frac{5}{3} + \frac{4}{3} \right) - \ln \left(\frac{\sqrt{10}}{3} + \frac{1}{3} \right) \\
&= \ln 3 - \ln(\sqrt{10} + 1) + \ln 3 \\
&= 2 \ln 3 - \ln(\sqrt{10} + 1)
\end{aligned}$$

8. $\int \frac{1}{x\sqrt{x^2 - 1}} dx$

Let $x = \sec \theta$, $dx = \sec \theta \tan \theta d\theta$ for $0 < \theta < \frac{\pi}{2}$

$$\begin{aligned}
\int \frac{1}{x\sqrt{x^2 - 1}} dx &= \int \frac{\sec \theta \tan \theta}{\sec \theta \sqrt{\sec^2 \theta - 1}} d\theta \\
&= \int \frac{\sec \theta \tan \theta}{\sec \theta \sqrt{\tan^2 \theta}} d\theta \\
&= \int 1 d\theta \\
&= \theta + C \\
&= \sec^{-1} x + C
\end{aligned}$$

9. $\int \frac{1}{\sqrt{1 - x^2}} dx$

Let $x = \sin \theta$, $dx = \cos \theta d\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\begin{aligned}
 \int \frac{1}{\sqrt{1-x^2}} dx &= \int \frac{\cos \theta}{\sqrt{1-\sin^2 \theta}} d\theta \\
 &= \int \frac{\cos \theta}{\cos \theta} d\theta \quad \left(\cos \theta > 0, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \right) \\
 &= \int 1 d\theta \\
 &= \theta + C \\
 &= \sin^{-1} x + C
 \end{aligned}$$

Section 8.7:

1.

$$\begin{aligned}
 \int_0^1 \frac{1}{\sqrt{x}} dx &= \lim_{b \rightarrow 0^+} \int_b^1 \frac{1}{\sqrt{x}} dx \\
 &= \lim_{b \rightarrow 0^+} 2\sqrt{x} \Big|_b^1 \\
 &= \lim_{b \rightarrow 0^+} (2 - 2\sqrt{b}) \\
 &= 2
 \end{aligned}$$

2.

$$\begin{aligned}
 \int_0^1 \frac{1}{\sqrt{1-x^2}} dx &= \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{\sqrt{1-x^2}} dx \\
 &= \lim_{b \rightarrow 1^-} \sin^{-1} x \Big|_0^b \\
 &= \lim_{b \rightarrow 1^-} (\sin^{-1}(b) - \sin^{-1}(0)) \\
 &= \frac{\pi}{2}
 \end{aligned}$$

3.

$$\begin{aligned}
 \int_1^\infty \frac{1}{x\sqrt{x^2-1}} dx &= \lim_{b \rightarrow 1^+} \int_b^2 \frac{1}{x\sqrt{x^2-1}} dx + \lim_{c \rightarrow \infty} \int_2^c \frac{1}{x\sqrt{x^2-1}} dx \\
 &= \lim_{b \rightarrow 1^+} \sec^{-1} x \Big|_b^2 + \lim_{c \rightarrow \infty} \sec^{-1} x \Big|_2^c \\
 &= \lim_{b \rightarrow 1^+} [\sec^{-1}(2) - \sec^{-1}(b)] + \lim_{c \rightarrow \infty} [\sec^{-1}(c) - \sec^{-1}(2)] \\
 &= \frac{\pi}{3} - 0 + \frac{\pi}{2} - \frac{\pi}{3} \\
 &= \frac{\pi}{2}
 \end{aligned}$$

4. $\int_0^\infty 2e^{-\theta} \sin \theta \, d\theta$

We will first consider the indefinite integral $\int 2e^{-\theta} \sin \theta \, d\theta$.

Let $u_1 = \sin \theta$, $du_1 = \cos \theta \, d\theta$ and $v_1 = -2e^{-\theta}$, $dv_1 = 2e^{-\theta} \, d\theta$.

Let $u_2 = \cos \theta$, $du_2 = -\sin \theta \, d\theta$ and $v_2 = -2e^{-\theta}$, $dv_2 = 2e^{-\theta} \, d\theta$.

$$\begin{aligned} \int 2e^{-\theta} \sin \theta \, d\theta &= u_1 v_1 - \int v_1 \, du_1 \\ &= -2e^{-\theta} \sin \theta + \int 2e^{-\theta} \sin \theta \, d\theta \\ &= -2e^{-\theta} \sin \theta + u_2 v_2 - \int v_2 \, du_2 \\ &= -2e^{-\theta} \sin \theta - 2e^{-\theta} \cos \theta - \int 2e^{-\theta} \sin \theta \, d\theta \end{aligned}$$

Let $I = \int 2e^{-\theta} \sin \theta \, d\theta$. Solving for I we get:

$$\begin{aligned} I = -2e^{-\theta} \sin \theta - 2e^{-\theta} \cos \theta - I &\Leftrightarrow 2I = -2e^{-\theta} \sin \theta - 2e^{-\theta} \cos \theta \\ &\Leftrightarrow I = -e^{-\theta} \sin \theta - e^{-\theta} \cos \theta \end{aligned}$$

So we have that $\int 2e^{-\theta} \sin \theta \, d\theta = -e^{-\theta} \sin \theta - e^{-\theta} \cos \theta + C$. Now let's consider the improper integral $\int_0^\infty 2e^{-\theta} \sin \theta \, d\theta$.

$$\begin{aligned} \int_0^\infty 2e^{-\theta} \sin \theta \, d\theta &= \lim_{b \rightarrow \infty} \int_0^b 2e^{-\theta} \sin \theta \, d\theta \\ &= \lim_{b \rightarrow \infty} \left(-e^{-\theta} \sin \theta - e^{-\theta} \cos \theta \right) \Big|_0^b \\ &= \lim_{b \rightarrow \infty} [(-e^{-b} \sin b - e^{-b} \cos b) - (-1)] \\ &= 1 \end{aligned}$$

5.

$$\begin{aligned} \int_{-\infty}^\infty 2xe^{-x^2} \, dx &= \lim_{b \rightarrow -\infty} \int_b^0 2xe^{-x^2} \, dx + \lim_{c \rightarrow \infty} \int_0^c 2xe^{-x^2} \, dx \\ &= \lim_{b \rightarrow -\infty} \left(-e^{-x^2} \right) \Big|_b^0 + \lim_{c \rightarrow \infty} \left(-e^{-x^2} \right) \Big|_0^c \\ &= \lim_{b \rightarrow -\infty} (-1 + e^{-b^2}) + \lim_{c \rightarrow \infty} (-e^{-c^2} + 1) \\ &= 0 \end{aligned}$$

6.

$$\begin{aligned}
 \int_0^1 x \ln x \, dx &= \lim_{b \rightarrow 0^+} \int_b^1 x \ln x \, dx \\
 &= \lim_{b \rightarrow 0^+} \left[\frac{1}{2} x^2 \ln x \Big|_b^1 - \int_b^1 \frac{1}{2} x \, dx \right] && (\text{Integration by Parts}) \\
 &= \lim_{b \rightarrow 0^+} \left(\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 \right) \Big|_b^1 \\
 &= \lim_{b \rightarrow 0^+} \left[\left(-\frac{1}{4} \right) - \left(\frac{1}{2} b^2 \ln b - \frac{1}{4} b^2 \right) \right] \\
 &= -\frac{1}{4} - \frac{1}{2} \left[\lim_{b \rightarrow 0^+} \frac{\ln b}{\left(\frac{1}{b^2} \right)} \right] \\
 &= -\frac{1}{4} - \frac{1}{2} \left[\lim_{b \rightarrow 0^+} \frac{\left(\frac{1}{b} \right)}{\left(\frac{-2}{b^3} \right)} \right] && (l'Hopital's Rule) \\
 &= -\frac{1}{4} + \frac{1}{4} \left[\lim_{b \rightarrow 0^+} b^2 \right] \\
 &= -\frac{1}{4}
 \end{aligned}$$

7.

$$\begin{aligned}
 \int_{-1}^4 \frac{1}{\sqrt{|x|}} \, dx &= \lim_{b \rightarrow 0^-} \int_{-1}^b \frac{1}{\sqrt{|x|}} \, dx + \lim_{c \rightarrow 0^+} \int_c^4 \frac{1}{\sqrt{|x|}} \, dx \\
 &= \lim_{b \rightarrow 0^-} \int_{-1}^b \frac{1}{\sqrt{-x}} \, dx + \lim_{c \rightarrow 0^+} \int_c^4 \frac{1}{\sqrt{x}} \, dx \\
 &= \lim_{b \rightarrow 0^-} -2\sqrt{-x} \Big|_{-1}^b + \lim_{c \rightarrow 0^+} 2\sqrt{x} \Big|_c^4 \\
 &= \lim_{b \rightarrow 0^-} (-2\sqrt{-b} + 2) + \lim_{c \rightarrow 0^+} (4 - 2\sqrt{c}) \\
 &= 6
 \end{aligned}$$

Section 11.2:

1. $x = \cos t, y = 2 + \sin t, \quad 0 \leq t \leq 2\pi, \quad x\text{-axis}$

$$\begin{aligned}
 \text{AREA} &= \int_0^{2\pi} 2\pi(2 + \sin t) \sqrt{(-\sin t)^2 + (\cos t)^2} dt \\
 &= 2\pi \int_0^{2\pi} (2 + \sin t) \sqrt{\sin^2 t + \cos^2 t} dt \\
 &= 2\pi \int_0^{2\pi} 2 + \sin t dt \\
 &= 2\pi (2t - \cos t) \Big|_0^{2\pi} \\
 &= 2\pi [(4\pi - 1) - (0 - 1)] \\
 &= 8\pi^2
 \end{aligned}$$

2. $x = \ln(\sec t + \tan t) - \sin t, y = \cos t, \quad 0 \leq t \leq \frac{\pi}{3}; \quad x\text{-axis}$

$$\begin{aligned}
 \text{AREA} &= \int_0^{\pi/3} 2\pi \cos t \sqrt{\left(\frac{\sec t \tan t + \sec^2 t}{\sec t + \tan t} - \cos t \right)^2 + (-\sin t)^2} dt \\
 &= 2\pi \int_0^{\pi/3} \cos t \sqrt{\left(\frac{(\sec t)(\tan t + \sec t)}{\sec t + \tan t} - \cos t \right)^2 + \sin^2 t} dt \\
 &= 2\pi \int_0^{\pi/3} \cos t \sqrt{(\sec t - \cos t)^2 + \sin^2 t} dt \\
 &= 2\pi \int_0^{\pi/3} \cos t \sqrt{\sec^2 t - 2 + \cos^2 t + \sin^2 t} dt \\
 &= 2\pi \int_0^{\pi/3} \cos t \sqrt{\sec^2 t - 1} dt \\
 &= 2\pi \int_0^{\pi/3} \cos t \sqrt{\tan^2 t} dt \\
 &= 2\pi \int_0^{\pi/3} \sin t dt \\
 &= -2\pi \cos t \Big|_0^{\pi/3} \\
 &= -2\pi \left(\frac{1}{2} - 1 \right) \\
 &= \pi
 \end{aligned}$$