

## Math 21B - Homework Set 8

### Section 8.4:

$$\begin{aligned}
 1. \quad \int \frac{x+4}{x^2+5x-6} dx &= \int \frac{x+4}{(x+6)(x-1)} dx \\
 \frac{x+4}{(x+6)(x-1)} &= \frac{A}{x+6} + \frac{B}{x-1} \quad \Leftrightarrow \quad x+4 = (A+B)x + (-A+6B) \\
 &\Leftrightarrow \quad \begin{cases} 1 = A+B \\ 4 = -A+6B \end{cases} \\
 &\Leftrightarrow \quad A = \frac{2}{7}, \quad B = \frac{5}{7}
 \end{aligned}$$

Therefore, going back to our integral we get:

$$\begin{aligned}
 \int \frac{x+4}{x^2+5x-6} dx &= \int \left( \frac{2}{7} \cdot \frac{1}{x+6} + \frac{5}{7} \cdot \frac{1}{x-1} \right) dx \\
 &= \frac{2}{7} \ln|x+6| + \frac{5}{7} \ln|x-1| + C
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int \frac{x+3}{2x^3-8x} dx &= \int \frac{x+3}{2x(x-2)(x+2)} dx \\
 \frac{x+3}{2x(x-2)(x+2)} &= \frac{A}{2x} + \frac{B}{x-2} + \frac{C}{x+2} \quad \Leftrightarrow \quad x+3 = (A+2B+2C)x^2 + (4C-4B)x - 4A \\
 &\Leftrightarrow \quad \begin{cases} 0 = A+2B+2C \\ 1 = 4C-4B \\ 3 = -4A \end{cases} \\
 &\Leftrightarrow \quad A = -\frac{3}{4}, \quad B = \frac{1}{16}, \quad C = \frac{5}{16}
 \end{aligned}$$

Therefore, going back to our integral we get:

$$\begin{aligned}
 \int \frac{x+3}{2x^3-8x} dx &= \int \left( -\frac{3}{4} \cdot \frac{1}{2x} + \frac{1}{16} \cdot \frac{1}{x-2} + \frac{5}{16} \cdot \frac{1}{x+2} \right) dx \\
 &= -\frac{3}{8} \ln|x| + \frac{1}{16} \ln|x-2| + \frac{5}{16} \ln|x+2| + C
 \end{aligned}$$

$$\begin{aligned}
3. \int_0^1 \frac{x^3}{x^2+2x+1} dx &= \int_0^1 \left( x-2 + \frac{3x+2}{x^2+2x+1} \right) dx \quad (\text{by long division}) \\
\frac{3x+2}{x^2+2x+1} &= \frac{A}{x+1} + \frac{B}{(x+1)^2} \quad \Leftrightarrow \quad 3x+2 = Ax + (A+B) \\
&\Leftrightarrow \quad \begin{cases} 3 = A \\ 2 = A+B \end{cases} \\
&\Leftrightarrow A = 3, \quad B = -1
\end{aligned}$$

Therefore, going back to our integral we get:

$$\begin{aligned}
\int_0^1 \frac{x^3}{x^2+2x+1} dx &= \int_0^1 \left( x-2 + \frac{3}{x+1} - \frac{1}{(x+1)^2} \right) dx \\
&= \left( \frac{x^2}{2} - 2x + 3 \ln|x+1| + \frac{1}{x+1} \right) \Big|_0^1 \\
&= \left( \frac{1}{2} - 2 + 3 \ln 2 + \frac{1}{2} \right) - (1) \\
&= 3 \ln 2 - 2
\end{aligned}$$

$$\begin{aligned}
4. \int_0^1 \frac{dx}{(x+1)(x^2+1)} \\
\frac{1}{(x+1)(x^2+1)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \quad \Leftrightarrow \quad 1 = (A+B)x^2 + (B+C)x + (C+A) \\
&\Leftrightarrow \quad \begin{cases} 0 = A+B \\ 0 = B+C \\ 1 = C+A \end{cases} \\
&\Leftrightarrow \quad A = \frac{1}{2}, \quad B = -\frac{1}{2}, \quad C = \frac{1}{2}
\end{aligned}$$

Therefore, going back to our integral we get:

$$\begin{aligned}
\int_0^1 \frac{dx}{(x+1)(x^2+1)} &= \int_0^1 \left( \frac{1}{2} \cdot \frac{1}{x+1} + \frac{1}{2} \cdot \frac{1-x}{x^2+1} \right) dx \\
&= \frac{1}{2} \int_0^1 \left( \frac{1}{x+1} + \frac{1}{x^2+1} - \frac{x}{x^2+1} \right) dx \\
&= \frac{1}{2} \left( \ln|x+1| + \tan^{-1} x - \frac{1}{2} \ln|x^2+1| \right) \Big|_0^1 \\
&= \frac{1}{2} \left( \ln 2 + \frac{\pi}{4} - \frac{\ln 2}{2} \right) \\
&= \frac{1}{4} \ln 2 + \frac{\pi}{8}
\end{aligned}$$

$$\begin{aligned}
5. \quad & \int_1^{\sqrt{3}} \frac{3t^2 + t + 4}{t^3 + t} dt \\
& \frac{3t^2 + t + 4}{t(t^2 + 1)} = \frac{A}{t} + \frac{Bx + C}{t^2 + 1} \quad \Leftrightarrow \quad 3t^2 + t + 4 = (A + B)t^2 + Ct + A \\
& \Leftrightarrow \quad \begin{cases} 3 = A + B \\ 1 = C \\ 4 = A \end{cases} \\
& \Leftrightarrow \quad A = 4, \quad B = -1, \quad C = 1
\end{aligned}$$

Therefore, going back to our integral we get:

$$\begin{aligned}
\int_1^{\sqrt{3}} \frac{3t^2 + t + 4}{t^3 + t} dx &= \int_1^{\sqrt{3}} \left( \frac{4}{t} + \frac{1-t}{t^2 + 1} \right) dt \\
&= \int_1^{\sqrt{3}} \left( \frac{4}{t} + \frac{1}{t^2 + 1} - \frac{t}{t^2 + 1} \right) dt \\
&= \left( 4 \ln |t| + \tan^{-1} t - \frac{1}{2} \ln |t^2 + 1| \right) \Big|_1^{\sqrt{3}} \\
&= \left( 4 \ln(\sqrt{3}) + \frac{\pi}{3} - \frac{1}{2} \ln 4 \right) - \left( \frac{\pi}{4} - \frac{1}{2} \ln 2 \right) \\
&= 2 \ln 3 + \frac{\pi}{3} - \ln 2 - \frac{\pi}{4} + \frac{1}{2} \ln 2 \\
&= 2 \ln 3 - \frac{1}{2} \ln 2 + \frac{\pi}{12}
\end{aligned}$$

$$\begin{aligned}
6. \quad & \int \frac{x^4}{x^2 - 1} dx = \int \left( x^2 + 1 + \frac{1}{x^2 - 1} \right) dx \\
& \frac{1}{x^2 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1} \quad \Leftrightarrow \quad 1 = (A + B)x + (A - B) \\
& \Leftrightarrow \quad \begin{cases} 0 = A + B \\ 1 = A - B \end{cases} \\
& \Leftrightarrow \quad A = \frac{1}{2}, \quad B = -\frac{1}{2}
\end{aligned}$$

Therefore, going back to our integral we get:

$$\begin{aligned}
\int \frac{x^4}{x^2 - 1} dx &= \int \left( x^2 + 1 + \frac{1}{2} \cdot \frac{1}{x - 1} - \frac{1}{2} \cdot \frac{1}{x + 1} \right) dx \\
&= \frac{1}{3} x^3 + x + \frac{1}{2} \ln |x - 1| - \frac{1}{2} \ln |x + 1| + C \\
&= \frac{1}{3} x^3 + x + \frac{1}{2} \ln \left| \frac{x - 1}{x + 1} \right| + C
\end{aligned}$$

**Section 8.7:**

1.

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}} dx &= \lim_{b \rightarrow 0^+} \int_b^1 \frac{1}{\sqrt{x}} dx \\ &= \lim_{b \rightarrow 0^+} 2\sqrt{x} \Big|_b^1 \\ &= \lim_{b \rightarrow 0^+} (2 - 2\sqrt{b}) \\ &= 2\end{aligned}$$

2.

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{1-x^2}} dx &= \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{\sqrt{1-x^2}} dx \\ &= \lim_{b \rightarrow 1^-} \sin^{-1} x \Big|_0^b \\ &= \lim_{b \rightarrow 1^-} (\sin^{-1}(b) - \sin^{-1}(0)) \\ &= \frac{\pi}{2}\end{aligned}$$

3.

$$\begin{aligned}\int_1^\infty \frac{1}{x\sqrt{x^2-1}} dx &= \lim_{b \rightarrow 1^+} \int_b^2 \frac{1}{x\sqrt{x^2-1}} dx + \lim_{c \rightarrow \infty} \int_2^c \frac{1}{x\sqrt{x^2-1}} dx \\ &= \lim_{b \rightarrow 1^+} \sec^{-1} x \Big|_b^2 + \lim_{c \rightarrow \infty} \sec^{-1} x \Big|_2^c \\ &= \lim_{b \rightarrow 1^+} [\sec^{-1}(2) - \sec^{-1}(b)] + \lim_{c \rightarrow \infty} [\sec^{-1}(c) - \sec^{-1}(2)] \\ &= \frac{\pi}{3} - 0 + \frac{\pi}{2} - \frac{\pi}{3} \\ &= \frac{\pi}{2}\end{aligned}$$

4.  $\int_0^\infty 2e^{-\theta} \sin \theta d\theta$

We will first consider the indefinite integral  $\int 2e^{-\theta} \sin \theta d\theta$ .

Let  $u_1 = \sin \theta$ ,  $du_1 = \cos \theta d\theta$  and  $v_1 = -2e^{-\theta}$ ,  $dv_1 = 2e^{-\theta} d\theta$ .

Let  $u_2 = \cos \theta$ ,  $du_2 = -\sin \theta d\theta$  and  $v_2 = -2e^{-\theta}$ ,  $dv_2 = 2e^{-\theta} d\theta$ .

$$\begin{aligned}
\int 2e^{-\theta} \sin \theta d\theta &= u_1 v_1 - \int v_1 du_1 \\
&= -2e^{-\theta} \sin \theta + \int 2e^{-\theta} \sin \theta d\theta \\
&= -2e^{-\theta} \sin \theta + u_2 v_2 - \int v_2 du_2 \\
&= -2e^{-\theta} \sin \theta - 2e^{-\theta} \cos \theta - \int 2e^{-\theta} \sin \theta d\theta
\end{aligned}$$

Let  $I = \int 2e^{-\theta} \sin \theta d\theta$ . Solving for  $I$  we get:

$$\begin{aligned}
I = -2e^{-\theta} \sin \theta - 2e^{-\theta} \cos \theta - I &\Leftrightarrow 2I = -2e^{-\theta} \sin \theta - 2e^{-\theta} \cos \theta \\
&\Leftrightarrow I = -e^{-\theta} \sin \theta - e^{-\theta} \cos \theta
\end{aligned}$$

So we have that  $\int 2e^{-\theta} \sin \theta d\theta = -e^{-\theta} \sin \theta - e^{-\theta} \cos \theta + C$ . Now let's consider the improper integral  $\int_0^{\infty} 2e^{-\theta} \sin \theta d\theta$ .

$$\begin{aligned}
\int_0^{\infty} 2e^{-\theta} \sin \theta d\theta &= \lim_{b \rightarrow \infty} \int_0^b 2e^{-\theta} \sin \theta d\theta \\
&= \lim_{b \rightarrow \infty} \left( -e^{-\theta} \sin \theta - e^{-\theta} \cos \theta \right) \Big|_0^b \\
&= \lim_{b \rightarrow \infty} [(-e^{-b} \sin b - e^{-b} \cos b) - (-1)] \\
&= 1
\end{aligned}$$

5.

$$\begin{aligned}
\int_{-\infty}^{\infty} 2xe^{-x^2} dx &= \lim_{b \rightarrow -\infty} \int_b^0 2xe^{-x^2} dx + \lim_{c \rightarrow \infty} \int_0^c 2xe^{-x^2} dx \\
&= \lim_{b \rightarrow -\infty} -e^{-x^2} \Big|_b^0 + \lim_{c \rightarrow \infty} -e^{-x^2} \Big|_0^c \\
&= \lim_{b \rightarrow -\infty} (-1 + e^{-b^2}) + \lim_{c \rightarrow \infty} (-e^{-c^2} + 1) \\
&= 0
\end{aligned}$$

6.

$$\begin{aligned}
 \int_0^1 x \ln x \, dx &= \lim_{b \rightarrow 0^+} \int_b^1 x \ln x \, dx \\
 &= \lim_{b \rightarrow 0^+} \left[ \frac{1}{2} x^2 \ln x \Big|_b^1 - \int_b^1 \frac{1}{2} x \, dx \right] && (Integration \, by \, Parts) \\
 &= \lim_{b \rightarrow 0^+} \left( \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 \right) \Big|_b^1 \\
 &= \lim_{b \rightarrow 0^+} \left[ \left( -\frac{1}{4} \right) - \left( \frac{1}{2} b^2 \ln b - \frac{1}{4} b^2 \right) \right] \\
 &= -\frac{1}{4} - \frac{1}{2} \left[ \lim_{b \rightarrow 0^+} \frac{\ln b}{\left( \frac{1}{b^2} \right)} \right] \\
 &= -\frac{1}{4} - \frac{1}{2} \left[ \lim_{b \rightarrow 0^+} \frac{\left( \frac{1}{b} \right)}{\left( \frac{-2}{b^3} \right)} \right] && (l'Hopital's \, Rule) \\
 &= -\frac{1}{4} + \frac{1}{4} \left[ \lim_{b \rightarrow 0^+} b^2 \right] \\
 &= -\frac{1}{4}
 \end{aligned}$$

7.

$$\begin{aligned}
 \int_{-1}^4 \frac{1}{\sqrt{|x|}} \, dx &= \lim_{b \rightarrow 0^-} \int_{-1}^b \frac{1}{\sqrt{|x|}} \, dx + \lim_{c \rightarrow 0^+} \int_c^4 \frac{1}{\sqrt{|x|}} \, dx \\
 &= \lim_{b \rightarrow 0^-} \int_{-1}^b \frac{1}{\sqrt{-x}} \, dx + \lim_{c \rightarrow 0^+} \int_c^4 \frac{1}{\sqrt{x}} \, dx \\
 &= \lim_{b \rightarrow 0^-} \left. -2\sqrt{-x} \right|_{-1}^b + \lim_{c \rightarrow 0^+} \left. 2\sqrt{x} \right|_c^4 \\
 &= \lim_{b \rightarrow 0^-} \left( -2\sqrt{-b} + 2 \right) + \lim_{c \rightarrow 0^+} \left( 4 - 2\sqrt{c} \right) \\
 &= 6
 \end{aligned}$$

**Section 11.2:**

1.  $x = \cos t, y = 2 + \sin t, \quad 0 \leq t \leq 2\pi, \quad x\text{-axis}$

$$\begin{aligned}
 \text{AREA} &= \int_0^{2\pi} 2\pi(2 + \sin t) \sqrt{(-\sin t)^2 + (\cos t)^2} dt \\
 &= 2\pi \int_0^{2\pi} (2 + \sin t) \sqrt{\sin^2 t + \cos^2 t} dt \\
 &= 2\pi \int_0^{2\pi} 2 + \sin t dt \\
 &= 2\pi (2t - \cos t) \Big|_0^{2\pi} \\
 &= 2\pi [(4\pi - 1) - (0 - 1)] \\
 &= 8\pi^2
 \end{aligned}$$

2.  $x = \ln(\sec t + \tan t) - \sin t, y = \cos t, \quad 0 \leq t \leq \frac{\pi}{3}; \quad x\text{-axis}$

$$\begin{aligned}
 \text{AREA} &= \int_0^{\pi/3} 2\pi \cos t \sqrt{\left( \frac{\sec t \tan t + \sec^2 t}{\sec t + \tan t} - \cos t \right)^2 + (-\sin t)^2} dt \\
 &= 2\pi \int_0^{\pi/3} \cos t \sqrt{\left( \frac{(\sec t)(\tan t + \sec t)}{\sec t + \tan t} - \cos t \right)^2 + \sin^2 t} dt \\
 &= 2\pi \int_0^{\pi/3} \cos t \sqrt{(\sec t - \cos t)^2 + \sin^2 t} dt \\
 &= 2\pi \int_0^{\pi/3} \cos t \sqrt{\sec^2 t - 2 + \cos^2 t + \sin^2 t} dt \\
 &= 2\pi \int_0^{\pi/3} \cos t \sqrt{\sec^2 t - 1} dt \\
 &= 2\pi \int_0^{\pi/3} \cos t \sqrt{\tan^2 t} dt \\
 &= 2\pi \int_0^{\pi/3} \sin t dt \\
 &= -2\pi \cos t \Big|_0^{\pi/3} \\
 &= -2\pi \left( \frac{1}{2} - 1 \right) \\
 &= \pi
 \end{aligned}$$