NAME(print in CAPITAL letters, first name first): \underline{Key}

NAME(sign): ________________________________

ID#: ________________________________

Instructions: There are four problems. Make sure that you have all 4 problems. You must show all your work to receive full credit. Do not simplify complicated expressions unless instructed to do so.

Points received:
1
2
3
4
TOTAL
1. (25 points.) A box has $2n$ balls, $n$ of which are black. Let $X_n$ be the number of black balls in $n$ samples without replacement from the box.

(a) Find $E(X_n)$.  
\[ X_n = I_1 + \cdots + I_n \]
\[ I_j = I(\text{$j$th ball is black}) \]
\[ E(I_j) = P(\text{$j$th ball is black}) = \frac{1}{2} \]
\[ \text{So } E(X_n) = E(I_1) + \cdots + E(I_n) = \frac{n}{2} \]

(b) Find $\text{var}(X_n)$.
\[ \text{Var}(X_n) = \sum_{j=1}^{n} \text{Var}(I_j) + \sum_{j \neq k} \text{Cov}(I_j, I_k) \]
\[ \text{Var}(I_j) = \frac{1}{2} \left( 1 - \frac{1}{2} \right) \text{ (it's Bernoulli) } \]
\[ \text{Cov}(I_j, I_k) = E(I_j I_k) - E(I_j) E(I_k) \]
\[ = P(\text{$j$th & $k$th are black}) - \left( \frac{1}{2} \right)^2 \]
\[ = \frac{1}{2} \cdot \frac{n-1}{2n-1} - \frac{1}{4} \]
\[ \text{So } \text{Var}(X_n) = \frac{n}{2} \left( 1 - \frac{1}{2} \right) + n(n-1) \left[ \frac{1}{2} \cdot \frac{n-1}{2n-1} - \frac{1}{4} \right] \]

(c) Find the limit in probability, as $n \to \infty$, of $\frac{X_n}{n}$. Explain your reasoning.

Above, the covariance terms are $< 0$, so
\[ \text{Var}(X_n) < n \left( 1 - \frac{1}{2} \right) < n \]
\[ \text{Var} \left( \frac{X_n}{n} \right) = \frac{1}{n^2} \text{Var}(X_n) \leq \frac{n}{n^2} \to 0 \]

And \[ E \left( \frac{X_n}{n} \right) \to \frac{1}{2} \]
\[ \text{So } \frac{X_n}{n} \to \frac{1}{2} \text{ in probability} \]
2. (25 points.) A pair $(X, Y)$ of random variables has the joint density

$$f(x, y) = \begin{cases} 
1 & \text{if } x \geq 0 \text{ and } 0 \leq y \leq e^{-x}; \\
0 & \text{otherwise.}
\end{cases}$$

(a) Find the conditional density of $Y$, given $X = x$. Do you recognize the distribution?

$$f_X(x) = \int_0^{e^{-x}} f(x, y) \, dy$$

$$= e^{-x} \int_0^{e^{-x}} dy = e^{-x}$$

$$f_Y(y \mid X = x) = \frac{f(x, y)}{f_X(x)} = e^x, \quad 0 \leq y \leq e^{-x}$$

Here $x$ is constant so this is the uniform density over $[0, e^{-x}]$

(b) Find $E(Y \mid X = x)$.

$$E(Y \mid X = x) = \frac{1}{2} e^{-x}$$

(Expectation of a uniform $[0, e^{-x}]$ random variable)
3. (25 points.) You start with $1 and make repeated bets of $1 on red in roulette, until you lose all your money. You win each bet with probability $p = \frac{2}{3}$. Find the expected length of your game (i.e., the number of bets until you lose all your money). You may assume that the expectation is finite without any justification.

Let $N$ be the number of bets and $x = EN$

Condition on the result of the first bet. You get

$$EN = (1-p) E(N | \text{lose}) + p E(N | \text{win})$$

$$E(N | \text{lose}) = 1$$

$$E(N | \text{win}) = 1 + 2EN$$

(since if you have $2, now you have to lose $1 twice)

So

$$x = (1-p) + p(1 + 2x)$$

or

$$x = \frac{1}{1-2p} = 19$$
4. (25 points.) Suppose that $X, X_1, X_2, \ldots$ are i.i.d., with a moment generating function $\phi_X$ that satisfies $\phi_X(t) \leq e^{t^2}$. Define $S_n = X_1 + X_2 + \ldots + X_n$.

(a) Find an upper bound on the moment generating function of $S_n$.

$$\phi_{S_n}(t) = \left[\phi_X(t)\right]^n \leq e^{(t+t^2)n}$$

(b) Use your answer from part (a) to find an upper bound on $P(S_n \geq 2n)$.

$$P(S_n \geq 2n) = P(e^{tS_n} \geq e^{t \cdot 2n})$$

Markov $\Rightarrow e^{-2tn} E(e^{tS_n}) \phi_{S_n}(t) \leq e^{-2tn} e^{(t+t^2)n} = e^{n(t^2-t)}$

To get the best bound use $t = \frac{1}{2}$ (to minimize $t^2-t$) and get

$$P(S_n \geq 2n) \leq e^{-n/4}$$

You can also get this bound using Theorem 10.2.