Math 21B - Homework Set 1

Section 5.1:

2. $f(x) = x^3$ between $x = 0$ and $x = 1$.

   a. Estimate using lower sum with two rectangles of equal width:

   If we want two rectangles of equal width, we will let $\Delta x = \frac{1}{2}$. The function $f(x)$ is increasing on $[0, 1]$, so the height of each rectangle is given by the value of f at its left endpoint.

$$\begin{aligned} f(0) &= 0 \\ f\left(\frac{1}{2}\right) &= \frac{1}{8} \end{aligned}$$

   Thus we get:

$$\begin{aligned} A &\approx 0 \cdot \frac{1}{2} + \frac{1}{8} \cdot \frac{1}{2} \\ &= \frac{1}{16} \end{aligned}$$

   b. Estimate using lower sum with four rectangles of equal width:

   We will let $\Delta x = \frac{1}{4}$ and the heighths of the rectangles are given by the value of f at their respective left endpoints.

$$\begin{aligned} f(0) &= 0 \\ f\left(\frac{1}{4}\right) &= \frac{1}{64} \\ f\left(\frac{1}{2}\right) &= \frac{1}{8} \\ f\left(\frac{3}{4}\right) &= \frac{27}{64} \end{aligned}$$

   Thus we get:

$$\begin{aligned} A &\approx 0 \cdot \frac{1}{4} + \frac{1}{64} \cdot \frac{1}{4} + \frac{1}{8} \cdot \frac{1}{4} + \frac{27}{64} \cdot \frac{1}{4} \\ &= \frac{36}{256} \\ &= \frac{9}{64} \end{aligned}$$

c. Estimate using upper sum with two rectangles of equal width:

We will let $\Delta x = \frac{1}{2}$ and the heighths of the rectangles are given by the value of f at their respective right endpoints.

$$\begin{aligned} f\left(\frac{1}{2}\right) &= \frac{1}{8} \\ f(1) &= 1 \end{aligned}$$

Thus we get:

$$\begin{aligned} A &\approx \frac{1}{8} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} \\ &= \frac{9}{16} \end{aligned}$$

d. Estimate using upper sum with four rectangles of equal width:

We will let $\Delta x = \frac{1}{4}$ and the heighths of the rectangles are given by the value of f at their respective right endpoints.

$$\begin{aligned} f\left(\frac{1}{4}\right) &= \frac{1}{64} \\ f\left(\frac{1}{2}\right) &= \frac{1}{8} \\ f\left(\frac{3}{4}\right) &= \frac{27}{64} \\ f(1) &= 1 \end{aligned}$$

Thus we get:

$$\begin{aligned} A &\approx \frac{1}{64} \cdot \frac{1}{4} + \frac{1}{8} \cdot \frac{1}{4} + \frac{27}{64} \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} \\ &= \frac{100}{256} \\ &= \frac{25}{64} \end{aligned}$$

3. $f(x) = \frac{1}{x}$ between $x = 1$ and $x = 5$.

a. Estimate using lower sum with two rectangles of equal width:

If we want two rectangles of equal width, we will let $\Delta x = 2$. The function $f(x)$ is decreasing on $[1, 5]$, so the height of each rectangle is given by the value of f at its right endpoint.

$$
\begin{aligned}
f(3) &= \frac{1}{3} \\
f(5) &= \frac{1}{5}
\end{aligned}
$$

Thus we get:

$$
\begin{aligned}
A &\approx \frac{1}{3} \cdot 2 + \frac{1}{5} \cdot 2 \\
&= \frac{16}{15}
\end{aligned}
$$

b. Estimate using lower sum with four rectangles of equal width:

We will let $\Delta x = 1$ and the heighths of the rectangles are given by the value of f at their respective right endpoints.

$$
\begin{aligned}
f(2) &= \frac{1}{2} \\
f(3) &= \frac{1}{3} \\
f(4) &= \frac{1}{4} \\
f(5) &= \frac{1}{5}
\end{aligned}
$$

Thus we get:

$$
\begin{aligned}
A &\approx \frac{1}{2} \cdot 1 + \frac{1}{3} \cdot 1 + \frac{1}{4} \cdot 1 + \frac{1}{5} \cdot 1 \\
&= \frac{77}{60}
\end{aligned}
$$

c. Estimate using upper sum with two rectangles of equal width:

We will let $\Delta x = 2$ and the heighths of the rectangles are given by the value of f at their respective left endpoints.

$$
\begin{aligned}
f(1) &= 1 \\
f(3) &= \frac{1}{3}
\end{aligned}
$$

Thus we get:

$$\begin{aligned} A &\approx 1 \cdot 2 + \frac{1}{3} \cdot 2 \\ &= \frac{8}{3} \end{aligned}$$

d. Estimate using upper sum with four rectangles of equal width:

We will let $\Delta x = 1$ and the heighths of the rectangles are given by the value of f at their respective left endpoints.

$$\begin{aligned} f(1) &= 1 \\ f(2) &= \frac{1}{2} \\ f(3) &= \frac{1}{3} \\ f(4) &= \frac{1}{4} \end{aligned}$$

Thus we get:

$$\begin{aligned} A &\approx 1 \cdot 1 + \frac{1}{2} \cdot 1 + \frac{1}{3} \cdot 1 + \frac{1}{4} \cdot 1 \\ &= \frac{25}{12} \end{aligned}$$

5. $f(x) = x^2$ between $x = 0$ and $x = 1$.

- 2 rectangles
  We will let $\Delta x = \frac{1}{2}$. To get the height of the rectangles we will use:

$$\begin{aligned} f\left(\frac{1}{4}\right) &= \frac{1}{16} \\ f\left(\frac{3}{4}\right) &= \frac{9}{16} \end{aligned}$$

Thus we get:

$$\begin{aligned} A &\approx \frac{1}{16} \cdot \frac{1}{2} + \frac{9}{16} \cdot \frac{1}{2} \\ &= \frac{10}{32} \\ &= \frac{5}{16} \end{aligned}$$

- 4 rectangles
  We will let $\Delta x = \frac{1}{4}$. To get the height of the rectangles we will use:

$$\begin{aligned} f\left(\frac{1}{8}\right) &= \frac{1}{64} \\ f\left(\frac{3}{8}\right) &= \frac{9}{64} \\ f\left(\frac{5}{8}\right) &= \frac{25}{64} \\ f\left(\frac{7}{8}\right) &= \frac{49}{64} \end{aligned}$$

  Thus we get:

$$\begin{aligned} A &\approx \frac{1}{64}\cdot\frac{1}{4} + \frac{9}{64}\cdot\frac{1}{4} + \frac{25}{64}\cdot\frac{1}{4} + \frac{1}{49}\cdot\frac{1}{4} \\ &= \frac{84}{256} \\ &= \frac{21}{64} \end{aligned}$$

15. $f(x) = x^3$ on $[0,2]$

We will let $\Delta x = \frac{1}{2}$. To get the height of the rectangles we will use:

$$\begin{aligned} f\left(\frac{1}{4}\right) &= \frac{1}{64} \\ f\left(\frac{3}{4}\right) &= \frac{27}{64} \\ f\left(\frac{5}{4}\right) &= \frac{125}{64} \\ f\left(\frac{7}{4}\right) &= \frac{343}{64} \end{aligned}$$

The area A under the graph:

$$\begin{aligned} A &\approx \frac{1}{64}\cdot\frac{1}{2} + \frac{27}{64}\cdot\frac{1}{2} + \frac{125}{64}\cdot\frac{1}{2} + \frac{343}{64}\cdot\frac{1}{2} \\ &= \frac{496}{128} \\ &= \frac{31}{8} \end{aligned}$$

So the average value of f on $[0,2]$ is approximately $\frac{1}{2}\cdot\frac{31}{8} = \frac{31}{16}$.

21. Inscribe a regular n-sided plygon inside a circle of radius 1 and compute the area of the polgon for the following values of n:

a. 4 (square)

If we inscribe a square in a circle of radius one, note that the diagonal of the square forms a diameter of the circle. Hence the diagonal has length 2, and the length of each of the legs of the square must be $\sqrt{2}$ (the Pythagorean Theorem tells us that $(\text{leg})^2 + (\text{leg})^2 = 2^2$). So the square has area:

$$A = \sqrt{2} \cdot \sqrt{2} = 2$$

b. 8 (octagon)

If we inscribe an octagon in a circle of radius of one, we can think of it as the union of 8 isosoceles triangles in the following manner: Draw a radius from the center of the circle to each of the vertices of the octagon. Each triangle is formed by one side of the octagon and the radii from its two vertices. The angles of a regular octagon are $\frac{3\pi}{4}$, so the base angles of our isosceles triangles are $\frac{3\pi}{8}$. Thus, using the trigonometric functions we conclude that each triangle has height $\sin\left(\frac{3\pi}{8}\right)$ and base $2\cos\left(\frac{3\pi}{8}\right)$. Thus the area of each triangle is:

$$\begin{aligned}
\text{Area of Triangle} &= \frac{1}{2} \cdot \sin\left(\frac{3\pi}{8}\right) \cdot 2\cos\left(\frac{3\pi}{8}\right) \\
&= \sin\left(\frac{3\pi}{8}\right)\cos\left(\frac{3\pi}{8}\right) \\
&= \frac{1}{2}\sin\left(\frac{3\pi}{4}\right) \\
&= \frac{1}{2}\frac{\sqrt{2}}{2} \\
&= \frac{\sqrt{2}}{4}
\end{aligned}$$

Since the octagon is the union of 8 such triangles, we get:

$$\begin{aligned}
A &= 8 \cdot \frac{\sqrt{2}}{4} \\
&= 2\sqrt{2}
\end{aligned}$$

c. 16

Similar to our method in (b), we think of the inscribed 16-gon as the union of 16 isosceles triangles. The angles of a regular 16-gon are

$\frac{7\pi}{8}$, so the base angles of our isosceles triangles are $\frac{7\pi}{16}$. Thus, using the trigonometric functions we conclude that each triangle has height $\sin\left(\frac{7\pi}{16}\right)$ and base $2\cos\left(\frac{7\pi}{16}\right)$. Thus the area of each triangle is:

$$\begin{aligned}
\text{Area of Triangle} &= \frac{1}{2}\cdot\sin\left(\frac{7\pi}{16}\right)\cdot 2\cos\left(\frac{7\pi}{16}\right)\\
&= \sin\left(\frac{7\pi}{16}\right)\cos\left(\frac{7\pi}{16}\right)\\
&= \frac{1}{2}\sin\left(\frac{7\pi}{8}\right)
\end{aligned}$$

Since the 16-gon is the union of 16 such triangles, we get:

$$\begin{aligned}
A &= 16\cdot\frac{1}{2}\sin\left(\frac{7\pi}{8}\right)\\
&= 8\sin\left(\frac{7\pi}{8}\right)
\end{aligned}$$

d. Compare the areas in parts (a), (b), and (c) with the area of the circle.

Each polygon in parts (a) - (c) has area less than the cirlce (π), but as the number of sides increases, the area approaches the area of the circle.

22. a. Inscribe a regular n-sided polygon inside a circle of radius 1 and compute the area of one of the n congruent triangles formed by drawing radii to the vertices of the polygon.

The angles of a regular n-gon are given by $(n-2)\pi$, so the base angles of our isosceles triangles are $\frac{(n-2)\pi}{2n}$. Thus, using the trig functions we conclude that each triangle has height $\sin\left(\frac{(n-2)\pi}{2n}\right)$ and base $2\cos\left(\frac{(n-2)\pi}{2n}\right)$. Thus the area of each triangle is:

$$\begin{aligned}
\text{Area of Triangle} &= \frac{1}{2}\cdot\sin\left(\frac{(n-2)\pi}{2n}\right)\cdot 2\cos\left(\frac{(n-2)\pi}{2n}\right)\\
&= \sin\left(\frac{(n-2)\pi}{2n}\right)\cos\left(\frac{(n-2)\pi}{2n}\right)\\
&= \frac{1}{2}\sin\left(\frac{(n-2)\pi}{n}\right)\\
&= \frac{1}{2}\sin\left(\frac{2\pi}{n}\right) && \textit{(property of sine)}
\end{aligned}$$

b. Compute the limit of the area of the inscribed polygon as $n \to \infty$.

Since the n-gon is the union of n of the triangles from part (a), we know:

$$\begin{aligned} A &= n \cdot \frac{1}{2} \sin\left(\frac{2\pi}{n}\right) \\ &= \frac{n}{2} \sin\left(\frac{2\pi}{n}\right) \end{aligned}$$

Thus if we want to know the area of the n-gon as $n \to \infty$, we get:

$$\begin{aligned} \lim_{n\to\infty} \text{Area of } n\text{-gon} &= \lim_{n\to\infty} \frac{n}{2} \sin\left(\frac{2\pi}{n}\right) \\ &= \lim_{n\to\infty} \pi \cdot \frac{\sin(2\pi/n)}{2\pi/n} \\ &= \lim_{x\to 0} \pi \cdot \frac{\sin(x)}{x} && \textit{(change of variables)} \\ &= \pi \end{aligned}$$

c. Repeat the computations in parts (a) and (b) for a circle of radius r.

i. The angles of a regular n-gon are given by $(n-2)\pi$, so the base angles of our isosceles triangles are $\frac{(n-2)\pi}{2n}$. Thus, using the trig functions we conclude that each triangle has height $r \sin\left(\frac{(n-2)\pi}{2n}\right)$ and base $2r \cos\left(\frac{(n-2)\pi}{2n}\right)$. Thus the area of each triangle is:

$$\begin{aligned} \text{Area of Triangle} &= \frac{1}{2} \cdot r \sin\left(\frac{(n-2)\pi}{2n}\right) \cdot 2r \cos\left(\frac{(n-2)\pi}{2n}\right) \\ &= r^2 \sin\left(\frac{(n-2)\pi}{2n}\right) \cos\left(\frac{(n-2)\pi}{2n}\right) \\ &= \frac{r^2}{2} \sin\left(\frac{(n-2)\pi}{n}\right) \\ &= \frac{r^2}{2} \sin\left(\frac{2\pi}{n}\right) && \textit{(property of sine)} \end{aligned}$$

ii. Since the n-gon is the union of n of the triangles from part (a), we know:

$$\begin{aligned} A &= n \cdot \frac{r^2}{2} \sin\left(\frac{2\pi}{n}\right) \\ &= \frac{r^2 n}{2} \sin\left(\frac{2\pi}{n}\right) \end{aligned}$$

Thus if we want to know the area of the n-gon as $n \to \infty$, we get:

$$\begin{aligned}
\lim_{n\to\infty} \text{Area of } n\text{-gon} &= \lim_{n\to\infty} \frac{r^2 n}{2} \sin\left(\frac{2\pi}{n}\right) \\
&= \lim_{n\to\infty} \pi r^2 \cdot \frac{\sin(2\pi/n)}{2\pi/n} \\
&= \lim_{x\to 0} \pi r^2 \cdot \frac{\sin(x)}{x} && \textit{(change of variables)} \\
&= \pi r^2
\end{aligned}$$

Section 5.2:

1.

$$\begin{aligned}
\sum_{k=1}^{2} \frac{6k}{k+1} &= \frac{6}{2} + \frac{12}{3} \\
&= 3 + 4 \\
&= 7
\end{aligned}$$

2.

$$\begin{aligned}
\sum_{k=1}^{3} \frac{k-1}{k} &= \frac{0}{1} + \frac{1}{2} + \frac{2}{3} \\
&= 0 + \frac{1}{2} + \frac{2}{3} \\
&= \frac{7}{6}
\end{aligned}$$

4.

$$\begin{aligned}
\sum_{k=1}^{5} \sin(k\pi) &= \sin(\pi) + \sin(2\pi) + \sin(3\pi) + \sin(4\pi) + \sin(5\pi) \\
&= 0
\end{aligned}$$

7. ALL

a. $\sum_{k=1}^{6} 2^{k-1} = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 1 + 2 + 4 + 8 + 16 + 32$

b. $\sum_{k=0}^{5} 2^k == 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 1 + 2 + 4 + 8 + 16 + 32$

c. $\sum_{k=-1}^{4} 2^{k+1} = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 1 + 2 + 4 + 8 + 16 + 32$

13. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\sum_{k=1}^{4}\frac{1}{2^k}$

14. $2+4+6+8+10=\sum_{k=1}^{5}2k$

15. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}=\sum_{k=1}^{5}\frac{(-1)^{k+1}}{k}$

17. Suppose that $\sum_{k=1}^{n}a_k=-5$ and $\sum_{k=1}^{n}b_k=6$. Find the values of:

a. $\sum_{k=1}^{n}3a_k=3\sum_{k=1}^{n}a_k=-15$

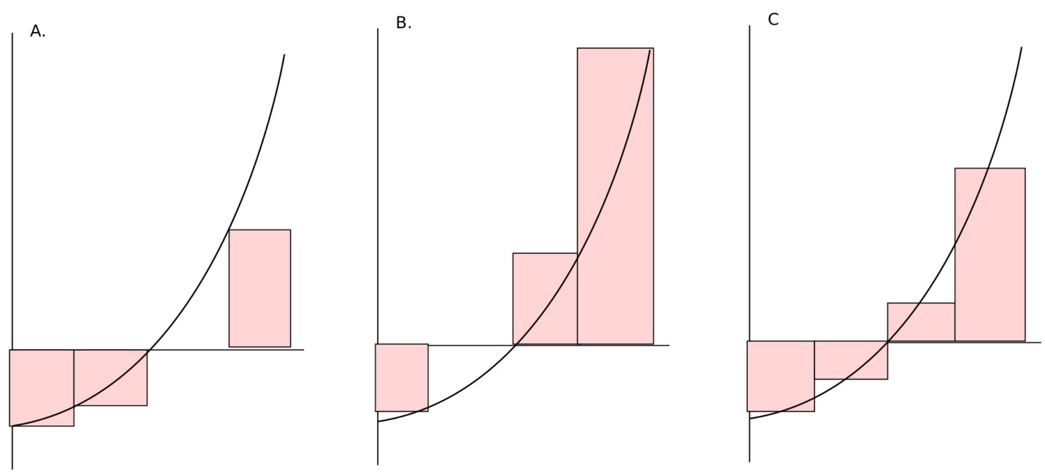
b. $\sum_{k=1}^{n}\frac{b_k}{6}=\frac{1}{6}\sum_{k=1}^{n}b_k=1$

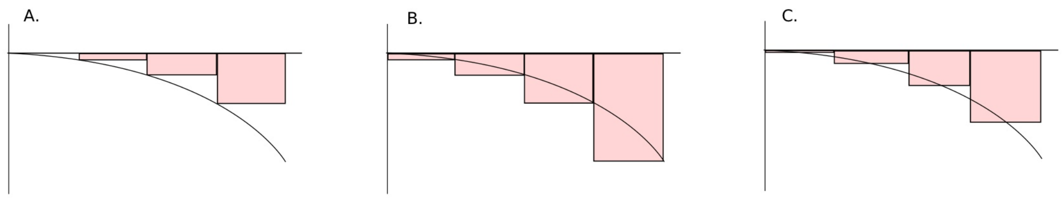
c. $\sum_{k=1}^{n}(a_k+b_k)=\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}b_k=1$

d. $\sum_{k=1}^{n}(a_k-b_k)=\sum_{k=1}^{n}a_k-\sum_{k=1}^{n}b_k=-11$

e. $\sum_{k=1}^{n}(b_k-2a_k)=\sum_{k=1}^{n}b_k-2\sum_{k=1}^{n}a_k=16$

29.



30. 

40. $f(x) = 3x + 2x^2$ over $[0, 1]$

If we want to take the upper sum using n equal subintervals, we will let $\Delta x = \frac{1}{n}$. Note that f is increasing on $[0, 1]$, so to get the upper sum, we will evaluate the function at the right endpoint of each subinterval.

$$
\begin{aligned}
A &\approx f\left(\frac{1}{n}\right) \cdot \frac{1}{n} + f\left(\frac{2}{n}\right) \cdot \frac{1}{n} + \ldots + f\left(\frac{n-1}{n}\right) \cdot \frac{1}{n} + f(1) \cdot \frac{1}{n} \\
&= \sum_{i=1}^{n} f\left(\frac{i}{n}\right) \cdot \frac{1}{n} \\
&= \sum_{i=1}^{n} \left(\frac{3i}{n} + \frac{2i^2}{n^2}\right) \cdot \frac{1}{n} \\
&= \frac{3}{n^2} \sum_{i=1}^{n} i + \frac{2}{n^3} \sum_{i=1}^{n} i^2 \\
&= \frac{3}{n^2}\left(\frac{n(n-1)}{2}\right) + \frac{2}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= \frac{3 + 3/n}{2} + \frac{2 + 3/n + 1/n^2}{3}
\end{aligned}
$$

Taking the limits as $n \to \infty$ gives:

$$
\lim_{n \to \infty} \frac{3 + 3/n}{2} + \frac{2 + 3/n + 1/n^2}{3} = \frac{3}{2} + \frac{2}{3} = \frac{13}{6}
$$