Math 21B-B - Homework Set 2

Section 5.3:

3. $\lim\limits_{\|P\|\to 0} \sum_{k=1}^{n} ({c_k}^2 - 3c_k)\Delta x_k$, where P is a partition of $[-7, 5]$.

$$\int_{-7}^{5} \left(x^2 - 3x^2\right)\, dx$$

6. $\lim\limits_{\|P\|\to 0} \sum_{k=1}^{n} \sqrt{4 - {c_k}^2}\Delta x_k$, where P is a partition of $[0, 1]$.

$$\int_{0}^{1} \sqrt{4 - x^2}\, dx$$

8. $\lim\limits_{\|P\|\to 0} \sum_{k=1}^{n} (\tan c_k)\Delta x_k$, where P is a partition of $[0, \pi/4]$.

$$\int_{0}^{\pi/4} \tan(x)\, dx$$

9. Suppose that f and g are integrable and that:

$$\int_1^2 f(x)\, dx = -4, \qquad \int_1^5 f(x)\, dx = 6, \qquad \int_1^5 g(x) = 8.$$

A. $\int_2^2 g(x)\, dx = 0$

Zero Width - Rule 2

B. $\int_5^1 g(x)\, dx = -\int_1^5 g(x)\, dx = -8$

Order of Integration - Rule 1

C. $\int_1^2 3f(x)\, dx = 3\int_1^2 f(x)\, dx = 3 \cdot -4 = -12.$

Constant Multiple - Rule 3

D. $\int_2^5 f(x)\, dx = \int_1^5 f(x)\, dx - \int_1^2 f(x)\, dx = 6 - (-4) = 10$

Additivity - Rule 5

E. $\int_1^5 [f(x) - g(x)]\, dx = \int_1^5 f(x)\, dx - \int_1^5 g(x)\, dx = 6 - 8 = -2$

SUM AND DIFFERENCE - RULE 4

F. $\int_1^5 [4f(x) - g(x)]\,dx = \int_1^5 4f(x)\,dx - \int_1^5 g(x)\,dx = 4\int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx = 4 \cdot 6 - 8 = 24 - 8 = 16$

CONSTANT MULTIPLE AND SUM/DIFFERENCE - RULES 3 AND 4

10. Suppose that f and h are integrable and that:

$$\int_1^9 f(x)\,dx = -1, \qquad \int_7^9 f(x)\,dx = 5, \qquad \int_7^9 h(x)\,dx = 4.$$

A. $\int_1^9 -2f(x)\,dx = -2\int_1^9 f(x)\,dx = -2 \cdot -1 = 2$

CONSTANT MULTIPLE - RULE 3

B. $\int_7^9 [f(x) + h(x)]\,dx = \int_7^9 f(x)\,dx + \int_7^9 h(x)\,dx = 5 + 4 = 9$

SUM AND DIFFERENCE - RULE 4

C. $\int_7^9 [2f(x) - 3h(x)]\,dx = \int_7^9 2f(x)\,dx - \int_7^9 3h(x)\,dx = 2\int_7^9 f(x)\,dx - 3\int_7^9 h(x)\,dx = 2 \cdot 5 - 3 \cdot 4 = 10 - 12 = -2$

CONSTANT MULTIPLE AND SUM/DIFFERENCE - RULES 3 AND 4

D. $\int_9^1 f(x)\,dx = -\int_1^9 f(x)\,dx = -(-1) = 1$

ORDER OF INTEGRATION - RULE 1

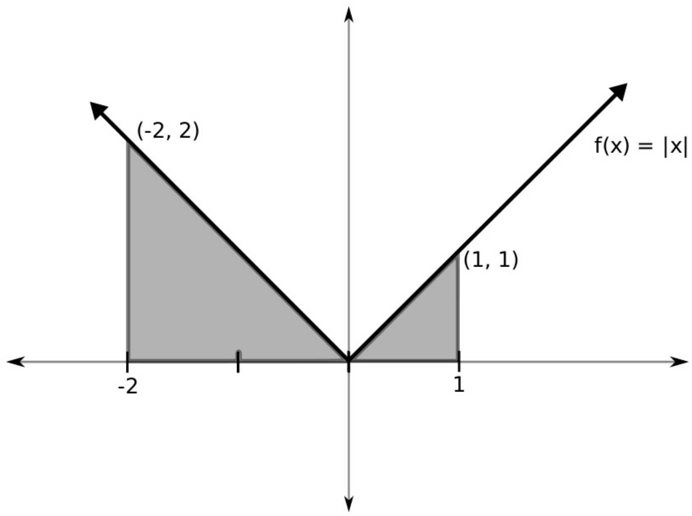
E. $\int_1^7 f(x)\,dx = \int_1^9 f(x)\,dx - \int_7^9 f(x)\,dx = -1 - 5 = -6$

ADDITIVITY - RULE 5

F. $\int_9^7 [h(x) - f(x)]\,dx = -\int_7^9 [h(x) - f(x)]\,dx = -\left[\int_7^9 h(x)\,dx - \int_7^9 f(x)\,dx\right] = -(4 - 5) = -(-1) = 1$

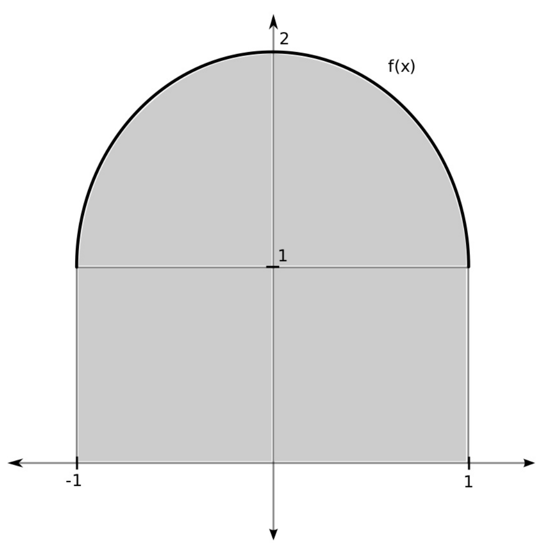
ORDER OF INTEGRATION AND SUM/DIFFERENCE - RULES 1 AND 4

19. $\int_{-2}^1 |x|\,dx$



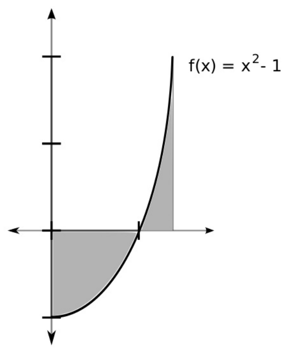
$$\begin{aligned}\int_{-2}^{1} |x|\,dx &= \text{Area of left triangle} + \text{Area of right triangle}\\ &= 1/2 \cdot 2 \cdot 2 + 1/2 \cdot 1 \cdot 1 = 2 + 1/2\\ &= 2 + 1/2\\ &= 5/2 \text{ square units}\end{aligned}$$

22. $\int_{-1}^{1} (1 + \sqrt{1 - x^2})\,dx$



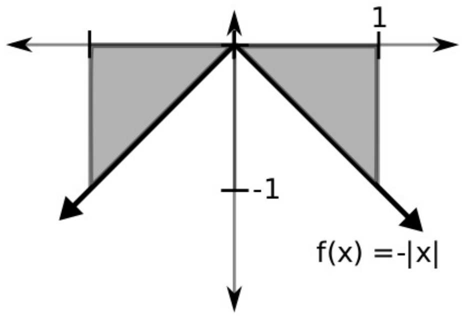
$$\begin{aligned}\int_{-1}^{1}(1+\sqrt{1-x^2})\,dx &= \text{Area of semicircle} + \text{Area of Rectangle}\\ &= 1/2\cdot\pi\cdot 1^2 + 1\cdot 2\\ &= \frac{\pi}{2} + 2 \text{ square units}\end{aligned}$$

55. $f(x) = x^2 - 1$ on $[0, \sqrt{3}]$



$$\begin{aligned}av(f) &= \left(\frac{1}{\sqrt{3}-0}\right)\int_0^{\sqrt{3}}(x^2-1)\,dx\\ &= \left(\frac{1}{\sqrt{3}}\right)\left[\int_0^{\sqrt{3}} x^2\,dx - \int_0^{\sqrt{3}} 1\,dx\right]\\ &= \left(\frac{1}{\sqrt{3}}\right)\left[\left(\frac{1}{3}\right)x^3\Big|_0^{\sqrt{3}} - x\Big|_0^{\sqrt{3}}\right]\\ &= \left(\frac{1}{\sqrt{3}}\right)\left[\left(\frac{1}{3}\right)(3\sqrt{3}-0) - (\sqrt{3}-0)\right]\\ &= \left(\frac{1}{3}\right)(\sqrt{3}-\sqrt{3})\\ &= 0\end{aligned}$$

62. $h(x) = -|x|$ on (A.) $[-1, 0]$, (B.) $[0, 1]$, (C.) $[-1, 1]$.



A.

$$\begin{aligned} av(f) &= \left(\frac{1}{0-(-1)}\right)\int_{-1}^{0} -|x|\,dx \\ &= (1)\int_{-1}^{0} -(-x)\,dx \\ &= \int_{-1}^{0} x\,dx \\ &= \frac{1}{2}x^2\Big|_{-1}^{0} \\ &= 0-\frac{1}{2} \\ &= -\frac{1}{2} \end{aligned}$$

B.

$$\begin{aligned} av(f) &= \left(\frac{1}{1-0}\right)\int_{0}^{1} -|x|\,dx \\ &= (1)\int_{0}^{1} -x\,dx \\ &= -\frac{1}{2}x^2\Big|_{0}^{1} \\ &= -\frac{1}{2} \end{aligned}$$

C.

$$\begin{aligned}
av(f) &= \left(\frac{1}{1-(-1)}\right)\int_{-1}^{1} -|x|\,dx \\
&= \left(\frac{1}{2}\right)\left[\int_{-1}^{0} -|x|\,dx + \int_{0}^{1} -|x|\,dx\right] \\
&= \left(\frac{1}{2}\right)\left[\int_{-1}^{0} -(-x)\,dx + \int_{0}^{1} -x\,dx\right] \\
&= \left(\frac{1}{2}\right)\left[\int_{-1}^{0} x\,dx + \int_{0}^{1} -x\,dx\right] \\
&= \left(\frac{1}{2}\right)\left[\frac{1}{2}x^2\Big|_{-1}^{0} - \frac{1}{2}x^2\Big|_{0}^{1}\right] \\
&= \left(\frac{1}{2}\right)\left[\left(0-\left(\frac{1}{2}\right)\right) - \left(\frac{1}{2}-0\right)\right] \\
&= \left(\frac{1}{2}\right)\left[-\frac{1}{2}-\frac{1}{2}\right] \\
&= \left(\frac{1}{2}\right)(-1) \\
&= -\frac{1}{2}
\end{aligned}$$

65. Use the Max-Min Inequality to find upper and lower bounds for the value of

$$\int_0^1 \frac{1}{1+x^2}\,dx$$

Notice that the integrand $f(x) = \frac{1}{1+x^2}$ is a decreasing function ($f'(x) = \frac{-2x}{(1+x^2)^2} \leq 0$ for $x \in [0,1]$). Therefore, on the interval f has a maximum at $x = 0$ and has a minimum at $x = 1$.

$$\begin{aligned}
\max f &= f(0) = 1 \\
\min f &= f(1) = \tfrac{1}{2}
\end{aligned}$$

Thus the Max-Min Inequality gives:

$$\tfrac{1}{2} \leq \int_0^1 \frac{1}{1+x^2}\,dx \leq 1$$

66. Use the Max-Min Inequality to find upper and lower bounds for the the value of

$$\int_0^{0.5} \frac{1}{1+x^2}\,dx \qquad \text{and} \qquad \int_{0.5}^{1} \frac{1}{1+x^2}\,dx$$

Using the observation (from 65) that $f(x) = \frac{1}{1+x^2}$ is decreasing on $[0, 0.5]$ and $[0.5, 1]$, we get the following two inequalities:

$$0.4 \le \int_0^{0.5} \frac{1}{1+x^2}\,dx \le 0.5$$
$$0.25 \le \int_{0.5}^{1} \frac{1}{1+x^2}\,dx \le 0.4$$

.

Therefore, by adding the two inequalities we get the new (and improved) estimate:

$$0.65 \le \int_0^1 \frac{1}{1+x^2}\,dx \le 0.9.$$

71. We know that $\sin x \le x$ for $x \ge 0$. Therefore we can get the upper bound:

$$\begin{aligned}\int_0^1 \sin x\,dx &\le \int_0^1 x\,dx \\ &= \frac{1}{2}x^2\Big|_0^1 \\ &= \frac{1}{2} - 0 \\ &= \frac{1}{2}\end{aligned}$$

Section 5.4:

1. $\int_{-2}^{0} (2x+5)\,dx = (x^2+5x)\big|_{-2}^{0} = (0+0) - (4-10) = 6$

2. $\int_{-3}^{4} \left(5 - \frac{x}{2}\right) dx = \left(5x - \frac{x^2}{4}\right)\Big|_{-3}^{4} = (20-4) - \left(-15 - \frac{9}{4}\right) = 16 + \frac{69}{4} = \frac{133}{4}$

9. $\int_0^{\pi} \sin x\,dx = -\cos x|_0^{\pi} = -\cos(\pi) + \cos(0) = 1 + 1 = 2$

11. $\displaystyle\int_0^{\pi/3} 2\sec^2 x\,dx = 2\tan x\,|_0^{\pi/3} = 2\tan(\pi/3) - 2\tan(0) = 2\cdot\sqrt{3} - 2\cdot 0 = 2\sqrt{3}$

19. $\displaystyle\int_1^{-1}(r+1)^2\,dr = -\int_{-1}^{1}(r^2+2r+1)\,dr = -\left[\left(\frac{1}{3}r^3+r^2+r\right)\Big|_{-1}^{1}\right] =$
$\displaystyle -\left[\left(\frac{1}{3}+1+1\right)-\left(-\frac{1}{3}+1-1\right)\right] = -\left[\frac{7}{3}+\frac{1}{3}\right] = -\frac{8}{3}$

28. $\displaystyle\int_1^2\left(\frac{1}{x}-e^{-x}\right)dx = \left(\ln x + e^{-x}\right)\big|_1^2 = (\ln 2 + e^{-2}) - (0 + e^{-1})$
$\ln 2 + \frac{1}{e^2} - \frac{1}{e}$

29. $\displaystyle\int_0^1 \frac{4}{1+x^2}\,dx = 4\arctan x\,|_0^1 = 4\arctan(1) - 4\arctan(0) = 4\cdot\frac{\pi}{4} - 4\cdot 0 = \pi$

30. $\displaystyle\int_2^5 \frac{x}{\sqrt{1+x^2}}\,dx = \int_2^5 x(1+x^2)^{-1/2}\,dx = \frac{1}{2}\cdot 2\cdot\sqrt{1+x^2}\Big|_2^5 = \sqrt{1+x^2}\Big|_2^5 =$
$\sqrt{26}-\sqrt{5}$

33. $\displaystyle\int_0^1 xe^{x^2}\,dx$

Notice that $\frac{d}{dx}x^2 = 2x$, and we have an x^2 in the exponent and an x in the product. Thus, keeping in mind the chain rule we get:

$f(x) = xe^{x^2} \qquad \Rightarrow \qquad F(x) = \frac{1}{2}e^{x^2}$

$$\int_0^1 xe^{x^2}\,dx = F(1) - F(0) = \frac{1}{2}e - \frac{1}{2} = \frac{1}{2}(e-1).$$

34. $\displaystyle\int_1^2 \frac{\ln x}{x}\,dx$

Notice that $\frac{d}{dx}\ln x = \frac{1}{x}$, and we have an $\ln x$ and a $\frac{1}{x}$. Thus, keeping in mind the chain rule we get:

$f(x) = \frac{\ln x}{x} \qquad \Rightarrow \qquad F(x) = \frac{1}{2}(\ln x)^2$

$$\int_1^2 \frac{\ln x}{x} = F(2) - F(1) = \frac{1}{2}(\ln 2)^2 - 0 = \frac{1}{2}(\ln 2)^2.$$

35. $\displaystyle\frac{d}{dx}\int_0^{\sqrt{x}} \cos t\,dt$

a.

$$\begin{aligned}\int_0^{\sqrt{x}} \cos t\, dt &= \sin t\,\big|_0^{\sqrt{x}} \\ &= \sin(\sqrt{x}) - \sin(0) \\ &= \sin(\sqrt{x})\end{aligned}$$

Therefore we get:

$$\begin{aligned}\frac{d}{dx}\int_0^{\sqrt{x}} \cos t\, dt &= \frac{d}{dx}\sin(\sqrt{x}) \\ &= \cos(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}\end{aligned}$$

b. Let $u = \sqrt{x}$

$$\begin{aligned}\frac{d}{dx}\int_0^{\sqrt{x}} \cos t\, dt &= \left(\frac{d}{du}\int_0^{u} \cos t\, dt\right) \cdot \frac{du}{dx} \\ &= \cos u \cdot \frac{du}{dx} \\ &= \cos(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}\end{aligned}$$

38. $\frac{d}{d\theta}\int_0^{\tan\theta} \sec^2 y\, dy$

a.

$$\begin{aligned}\int_0^{\tan\theta} \sec^2 y\, dy &= \tan y\,\big|_0^{\tan\theta} \\ &= \tan(\tan\theta) - \tan(0) \\ &= \tan(\tan\theta)\end{aligned}$$

Therefore we get:

$$\begin{aligned}\frac{d}{d\theta}\int_0^{\tan\theta} \sec^2 y\, dy &= \frac{d}{d\theta}\tan(\tan\theta) \\ &= \sec^2(\tan\theta) \cdot \sec^2\theta\end{aligned}$$

b. Let $u = \tan\theta$

$$\begin{aligned}\frac{d}{d\theta}\int_0^{\tan\theta} \sec^2 y\, dy &= \left(\frac{d}{du}\int_0^{u} \sec^2 y\, dy\right) \cdot \frac{du}{d\theta} \\ &= \sec^2 u \cdot \frac{du}{d\theta} \\ &= \sec^2(\tan\theta) \cdot \sec^2\theta\end{aligned}$$

43. $y = \int_{\sqrt{x}}^{0} \sin\left(t^2\right)\, dt = -\int_{0}^{\sqrt{x}} \sin\left(t^2\right)\, dt$

Let $u = \sqrt{x}$. Then we get:

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\sin(u^2) \cdot \frac{du}{dx} = -\sin(x) \cdot \frac{1}{x\sqrt{x}} = -\frac{\sin(x)}{2\sqrt{x}}$

51. $y = -x^2 - 2x \quad \text{where} \quad -3 \leq x \leq 2$

First, notice that $y = -x^2 - 2x = -x(x+2)$. Therefore in the interval $[-3, 2]$, the function has zeroes at $x = 0$ and $x = -2$.

$$\begin{aligned}
\text{Area} &= -\int_{-3}^{-2} \left(-x^2 - 2x\right) dx + \int_{-2}^{0} \left(-x^2 - 2x\right) dx - \int_{0}^{2} \left(-x^2 - 2x\right) dx \\
&= -\left(-\frac{1}{3}x^3 - x^2\right)\Big|_{-3}^{-2} + \left(-\frac{1}{3}x^3 - x^2\right)\Big|_{-2}^{0} - \left(-\frac{1}{3}x^3 - x^2\right)\Big|_{0}^{2} \\
&= -\left[\left(\frac{8}{3} - 4\right) - (9 - 9)\right] + \left[(0 - 0) - \left(\frac{8}{3} - 4\right)\right] - \left[\left(-\frac{8}{3} - 4\right) - (0 - 0)\right] \\
&= \frac{4}{3} + \frac{4}{3} + \frac{20}{3} \\
&= \frac{28}{3}
\end{aligned}$$

52. $y = 3x^2 - 3 \quad \text{where} \quad -2 \leq x \leq 2$

First notice that $y = 3(x^2 - 1) = 3(x-1)(x+1)$. Therefore in the interval $[-2, 2]$, the function has zeroes at $x = \pm 1$.

$$\begin{aligned}
\text{Area} &= \int_{-2}^{-1} \left(3x^2 - 3\right) dx - \int_{-1}^{1} \left(3x^2 - 2\right) dx + \int_{1}^{2} \left(3x^2 - 3\right) dx \\
&= \left(x^3 - 3x\right)\Big|_{-2}^{-1} - \left(x^3 - 3x\right)\Big|_{-1}^{1} + \left(x^3 - 3x\right)\Big|_{1}^{2} \\
&= [(-1 + 3) - (-8 + 6)] - [(1 - 3) - (-1 + 3)] + [(8 - 6) - (1 - 3)] \\
&= 4 + 4 + 4 \\
&= 12
\end{aligned}$$

56. $y = x^{1/3} - x \quad \text{where} \quad -1 \leq x \leq 8$

First notice that $y = x^{1/3} - x = x^{1/3}\left(1 - x^{1/3}\right)\left(1 + x^{1/3}\right)$, so the on $[-1, 8]$ the function has zeroes at $x = 0, \pm 1$.

$$
\begin{aligned}
\text{Area} &= -\int_{-1}^{0}\left(x^{1/3}-x\right)\,dx+\int_{0}^{1}\left(x^{1/3}-x\right)\,dx-\int_{1}^{8}\left(x^{1/3}-x\right)\,dx \\
&= -\left(\frac{3}{4}x^{4/3}-\frac{1}{2}x^{2}\right)\Big|_{-1}^{0}+\left(\frac{3}{4}x^{4/3}-\frac{1}{2}x^{2}\right)\Big|_{0}^{1}-\left(\frac{3}{4}x^{4/3}-\frac{1}{2}x^{2}\right)\Big|_{1}^{8} \\
&= -\left[(0-0)-\left(\frac{3}{4}-\frac{1}{2}\right)\right]+\left[\left(\frac{3}{4}-\frac{1}{2}\right)-(0-0)\right]-\left[(12-32)-\left(\frac{3}{4}-\frac{1}{2}\right)\right] \\
&= \frac{1}{4}+\frac{1}{4}+\frac{81}{4} \\
&= \frac{83}{4}
\end{aligned}
$$

73. a. $v=\frac{ds}{dt}=\frac{d}{dt}\int_0^t f(x)\,dx=f(t)$.

Therefore $v(5)=f(5)=2$ m/s.

b. Recall that the acceleration is the rate of change of the velocity, so $a=\frac{df}{dt}$. Since f has a negative slope at $t=5$, the acceleration is negative at $t=5$.

c. Since $s=\displaystyle\int_0^t f(x)\,dx$, we know that the position at $t=3$ can be represented as the area under the graph on $[0,3]$. Notice that on this interval, f is a straight line and so the value of the integral is just the area of a triangle formed by $f(x)$, the x-axis and $x=3$.

Therefore $s(3)=\frac{1}{2}\cdot 3\cdot 3=\frac{9}{2}$

d. We want to know when s has a maximum and we will determine this by looking at $f(x)$, which is the derivative of the s. We know f is positive on $[0,6]$ and negative on $[6,9]$, which means that s goes from increasing to decreasing at $t=6$. By the first derivative test, we know that s must have relative max at $t=6$.

e. At $x\approx 4$ and $x\approx 7$ since $s''(x)=f'(x)$ is zero at these points.

f. The particle is moving towards the origin when it has negative velocity $\rightarrow f$ is negative. Hence it is moving towards the origin on $[6,9]$. Similarly the particle is moving away from the origin when f is positive, hence on $[0,6]$.

g. Judging by the graph of f, we know that the particle is to the right of the origin (on the positive side) because the $s(9)=\displaystyle\int_0^9 f(x)\,dx>0$ (the area above the x-axis is greater than the area below it).

76. Find $\lim_{x\to 0} \frac{1}{x^3} \int_0^x \frac{t^2}{t^4+1}\, dt$

Notice that $\lim_{x\to 0} \int_0^x \frac{t^2}{t^4+1}\, dt = 0$ by the rules of integrals and $\lim_{x\to 0} x^3 = 0$. Thus if you think of the expression inside the limit as a fraction, we have that both the numerator and the denominator go to 0. Hence we can apply l'Hôpital's Rule to the limit and get:

$$\begin{aligned} \lim_{x\to 0} \frac{1}{x^3} \int_0^x \frac{t^2}{t^4+1}\, dt &= \lim_{x\to 0} \left[\left(\int_0^x \frac{t^2}{t^4+1}\, dt \right) / x^3 \right] \\ &= \lim_{x\to 0} \left[\left(\frac{x^2}{x^4+1} \right) / \left(3x^2\right) \right] \\ &= \lim_{x\to 0} \frac{1}{3\left(x^4+1\right)} \\ &= \frac{1}{3} \end{aligned}$$

81. Suppose $f'(x) \geq 0$ for all values of x, and that $f(1) = 0$. Define:

$$g(x) = \int_0^x f(t)\, dt$$

a. True. It follows from the FTC, part I.

b. True. Differentiable $\Rightarrow$ Continuous.

c. True. We know that $g'(1) = f(1) = 0$, so the tangent line at $x = 0$ is horizontal.

d. False. We know that f crosses the x-axis at $x = 1$ and that f is always increasing. Therefore $g'(x) = f(x)$ is negative for $x < 1$ and $g'(x) = f(x)$ is positive for $x > 1$. By the first derivative test, this is a minimum, not a max.

   Note that you could also use the fact that $g'(1) = 0$ and $g''(1) > 0$

e. True. See (d.)

f. False. $g''(x) = f'(x) > 0$, and hence does not change sign.

g. True. $g'(x) = f(x)$ and we know that the $f(1) = 0$ and is increasing.