Math 21B-B - Homework Set 4

Section 5.6:

3. a. $\int_0^{\frac{\pi}{4}} \tan x \sec^2 x \, dx$

Let $u = \tan x, \quad du = \sec^2 x \, dx$.

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \tan x \sec^2 x \, dx &= \int_{\tan 0}^{\tan \frac{\pi}{4}} u \, du \\
&= \int_0^1 u \, du \\
&= \frac{1}{2}u^2 \Big|_0^1 \\
&= \frac{1}{2}(1)^2 - \frac{1}{2}(0)^2 \\
&= \frac{1}{2}
\end{aligned}$$

b. $\int_{-\frac{\pi}{4}}^{0} \tan x \sec^2 x \, dx$

$$\begin{aligned}
\int_{-\frac{\pi}{4}}^{0} \tan x \sec^2 x \, dx &= \int_{\tan -\frac{\pi}{4}}^{\tan 0} u \, du \\
&= \int_{-1}^{0} u \, du \\
&= \frac{1}{2}u^2 \, du \\
&= \frac{1}{2}(0)^2 - \frac{1}{2}(-1)^2 \\
&= -\frac{1}{2}
\end{aligned}$$

6. a. $\int_0^{\sqrt{7}} t \left(t^2 + 1\right)^{1/3} \, dt$

Let $u = t^2 + 1, \quad du = 2t \, dt$.

$$\int_0^{\sqrt{7}} t\left(t^2+1\right)^{1/3}\,dt = \int_1^8 \frac{1}{2}u^{1/3}\,dt$$
$$= \frac{3}{8}u^{4/3}\Big|_1^8$$
$$= \frac{3}{8}\cdot 8^{4/3} - \frac{3}{8}\cdot 1^{4/3}$$
$$= 6 - \frac{3}{8}$$
$$= \frac{45}{8}$$

b. $\int_{-\sqrt{7}}^{0} t\left(t^2+1\right)^{1/3}\,dt$

$$\int_{-\sqrt{7}}^{0} t\left(t^2+1\right)^{1/3}\,dt = \int_8^0 \frac{1}{2}u^{1/3}\,du$$
$$= -\int_0^8 \frac{1}{2}u^{1/3}\,du$$
$$= -\frac{45}{8} \qquad \text{by (a.)}$$

10. a. $\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$

Let $u = x^4+9, \quad du = 4x^3\,dx$

$$\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx = \int_9^{10} \frac{1}{4\sqrt{u}}\,du$$
$$= \int_9^{10} \frac{1}{4}\,u^{-1/2}\,du$$
$$= \frac{1}{4}\cdot 2\,u^{1/2}\Big|_9^{10}$$
$$= \frac{1}{2}\sqrt{10} - \frac{1}{2}\sqrt{9}$$
$$= \frac{1}{2}\sqrt{10} - \frac{3}{2}$$

b. $\int_{-1}^{0} \frac{x^3}{\sqrt{x^4+9}}\,dx$

$$\int_{-1}^{0} \frac{x^3}{\sqrt{x^4+9}}\,dx = \int_{10}^{9} \frac{1}{4\sqrt{u}}\,du$$
$$= -\int_{9}^{10} \frac{1}{4\sqrt{u}}\,du$$
$$= \frac{3}{2} - \frac{1}{2}\sqrt{10} \qquad \text{by (b.)}$$

23. a. $\displaystyle\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta}\,\cos^2\left(\theta^{3/2}\right)\,d\theta$

Let $u = \theta^{3/2}$, $\quad du = \frac{3}{2}\,\theta^{1/2} = \frac{3}{2}\sqrt{\theta}\,d\theta$

$$\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta}\,\cos^2\left(\theta^{3/2}\right)\,d\theta = \int_0^{\pi} \frac{2}{3}\cos^2 u\,du$$
$$= \int_0^{\pi} \frac{2}{3}\left[\frac{1}{2} + \frac{1}{2}\cos(2u)\right]\,du$$
$$= \int_0^{\pi} \left[\frac{1}{3} + \frac{1}{3}\cos(2u)\right]\,du$$
$$= \left[\frac{u}{3} + \frac{1}{3}\cdot\frac{1}{2}\sin(2u)\right]\Bigg|_0^{\pi}$$
$$= \left[\frac{\pi}{3} + 0\right] - [0+0]$$
$$= \frac{\pi}{3}$$

b. $\displaystyle\int_{-1}^{-1/2} t^{-2}\sin^2\left(1+\frac{1}{t}\right)\,dt$

Let $u = 1 + \frac{1}{t}, \quad du = -\frac{1}{t^2}\, dt$

$$\begin{aligned}
\int_{-1}^{-1/2} t^{-2} \sin^2\left(1 + \frac{1}{t}\right) dt &= \int_0^{-1} -\sin^2 u \, du \\
&= \int_{-1}^{0} \sin^2 u \, du \\
&= \int_{-1}^{0} \left[\frac{1}{2} - \frac{1}{2}\cos(2u)\right] du \\
&= \left[\frac{u}{2} - \frac{1}{2} \cdot \frac{1}{2}\sin(2u)\right]\Big|_{-1}^{0} \\
&= (0 - 0) - \left(-\frac{1}{2} - \frac{1}{4}\sin(-2)\right) \\
&= \frac{1}{2} - \frac{1}{4}\sin(2)
\end{aligned}$$

30. $\displaystyle\int_2^4 \frac{dx}{x \ln x}$

Let $u = \ln x, \quad du = \frac{1}{x}\, dx$

$$\begin{aligned}
\int_2^4 \frac{dx}{x \ln x} &= \int_{\ln 2}^{\ln 4} \frac{du}{u} \\
&= \ln u\big|_{\ln 2}^{\ln 4} \\
&= \ln(\ln 4) - \ln(\ln 2) \\
&= \ln\left(\frac{\ln 4}{\ln 2}\right) \\
&= \ln\left(\frac{2\ln 2}{\ln 2}\right) \\
&= \ln 2
\end{aligned}$$

39. $\displaystyle\int_0^{\ln\sqrt{3}} \frac{e^x\, dx}{1 + e^{2x}}$

Let $u = e^x, \quad du = e^x\, dx$

$$\begin{aligned}
\int_0^{\ln\sqrt{3}} \frac{e^x\, dx}{1 + e^{2x}} &= \int_1^{\sqrt{3}} \frac{du}{1 + u^2} \\
&= \tan^{-1} u\big|_1^{\sqrt{3}} \\
&= \frac{\pi}{3} - \frac{\pi}{4} \\
&= \frac{\pi}{12}
\end{aligned}$$

47. $y = x\sqrt{4-x^2}$

We note that the function is negative on $[-2,0]$ and positive on $[0,2]$. Therefore, we get:

$$\text{Area} = -\int_{-2}^{0} x\sqrt{4-x^2}\,dx + \int_{0}^{2} x\sqrt{4-x^2}\,dx$$

Let $u = 4-x^2, \quad du = -2x\,dx.$

$$\begin{aligned}
\text{Area} &= -\int_{-2}^{0} x\sqrt{4-x^2}\,dx + \int_{0}^{2} x\sqrt{4-x^2}\,dx \\
&= -\int_{0}^{4} -\frac{1}{2}u^{1/2}\,du + \int_{4}^{0} -\frac{1}{2}u^{1/2}\,du \\
&= \int_{0}^{4} u^{1/2}\,du \\
&= \frac{2}{3}u^{3/2}\Big|_{0}^{4} \\
&= \frac{2}{3}\cdot 8 - \frac{2}{3}\cdot 0 \\
&= \frac{16}{3}
\end{aligned}$$

52. $y = \frac{1}{2}\sec^2 t$ and $y = -4\sin^2 t$

We note that the first function is positive on $[-\frac{\pi}{3}, \frac{\pi}{3}]$ and the second function is negative on this interval. Therefore, we get:

$$\begin{aligned}
\text{Area} &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}\sec^2 t + 4\sin^2 t\,dt \\
&= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}\sec^2 t + 4\left(\frac{1}{2} - \frac{1}{2}\cos(2t)\right)\,dt \\
&= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}\sec^2 t + 2 - 2\cos(2t)\,dt \\
&= \frac{1}{2}\tan t + 2t - \sin(2t)\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \\
&= \left(\frac{\sqrt{3}}{2} + \frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) - \left(-\frac{\sqrt{3}}{2} - \frac{2\pi}{3} + \frac{\sqrt{3}}{2}\right) \\
&= \frac{4\pi}{3}
\end{aligned}$$

55. $x = 2y^2 - 2y$ and $x = 12y^2 - 12y^3$

We will take the integral with respect to y. We not that the first function is negative (negative x-values) on $[0, 1]$, while the second function is positive. Therefore, we get:

$$\begin{aligned} \text{Area} &= \int_0^1 \left[-\left(2y^2 - 2y\right) + \left(12y^2 - 12y^3\right)\right] \, dy \\ &= \int_0^1 -12y^3 + 10y^2 + 2y \, dy \\ &= -3y^4 + \frac{10}{3}y^3 + y^2 \Big|_0^1 \\ &= \left(-3 + \frac{10}{3} + 1\right) - (0 + 0 + 0) \\ &= \frac{4}{3} \end{aligned}$$

58. $y = x^2$ and $x + y = 2$

We can integrate this with respect to either variable (x or y). For practice, let's set it up both ways:

wrt x: We want to find the area of the region under $y = x^2$ on $[0, 1]$ and under $y = 2 - x$ on $[1, 2]$. Therefore, we get the sum:

$$\begin{aligned} \text{Area} &= \int_0^1 x^2 \, dx + \int_1^2 2 - x \, dx \\ &= \frac{1}{3}x^3 \Big|_0^1 + 2x - \frac{1}{2}x^2 \Big|_1^2 \\ &= \left[\frac{1}{3} - 0\right] + \left[(4 - 2) - \left(2 - \frac{1}{2}\right)\right] \\ &= \frac{5}{6} \end{aligned}$$

wrt y: We want to find the area of (region bounded by $x = 2 - y$ and the y-axis on $[0, 1]$) - (region bounded by $x = \sqrt{y}$ and the y-axis on $[0, 1]$).

Therefore we get the integral:

$$\begin{aligned}\text{Area} &= \int_0^1 2 - y - \sqrt{y}\,dy \\ &= 2y - \frac{1}{2}y^2 - \frac{2}{3}y^{\frac{3}{2}}\Big|_0^1 \\ &= \left(2 - \frac{1}{2} - \frac{2}{3}\right) - (0 - 0 - 0) \\ &= \frac{5}{6}\end{aligned}$$

64. $y = 2x - x^2$ and $y = -3$

We first find the intersection points of the two functions:

$$\begin{aligned}2x - x^2 = -3 \quad &\Leftrightarrow \quad 0 = x^2 - 2x - 3 \\ &\Leftrightarrow \quad 0 = (x-3)(x+1) \\ &\Leftrightarrow \quad x = 3 \text{ or } x = -1\end{aligned}$$

We know $2x - x^2 \geq -3$ on $[-1, 3]$. Therefore we get the following integral:

$$\begin{aligned}\text{Area} &= \int_{-1}^3 \left(2x - x^2\right) - (-3)\,dx \\ &= \int_{-1}^3 -x^2 + 2x + 3\,dx \\ &= -\frac{1}{3}x^3 + x^2 + 3x\Big|_{-1}^3 \\ &= (-9 + 9 + 9) - \left(\frac{1}{3} + 1 - 3\right) \\ &= 9 + \frac{5}{3} \\ &= \frac{32}{3}\end{aligned}$$

71. $y = \sqrt{|x|}$ and $5y = x + 6$ (How many intersection points are there?)

We first find the intersection points of the two functions:

$$\begin{aligned}5\sqrt{|x|} = x + 6 \quad &\Leftrightarrow \quad 25|x| = x^2 + 12x + 36 \\ &\Leftrightarrow \quad 0 = x^2 + 12x - 25|x| + 36\end{aligned}$$

Therefore we have two cases to deal with:

$x \geq 0$

$$0 = x^2 + 12x - 25|x| + 36 \quad \Leftrightarrow \quad 0 = x^2 - 13x + 36$$
$$\Leftrightarrow \quad 0 = (x-4)(x-9)$$
$$\Leftrightarrow \quad x = 4 \quad \text{or} \quad x = 9$$

$x \leq 0$

$$0 = x^2 + 12x - 25|x| + 36 \quad \Leftrightarrow \quad 0 = x^2 + 37x + 36$$
$$\Leftrightarrow \quad 0 = (x+1)(x+36)$$
$$\Leftrightarrow \quad x = -1 \quad \text{or} \quad x = -36$$

Notice: $x = -36$ is not an intersection point (you can check this). You have to be careful when you are dealing with square roots. In solving we changed from $y = \sqrt{|x|}$ to $y^2 = |x|$. In the first expression $-\infty \leq x \leq \infty$ and $0 \leq y \leq \infty$ while in the second expression $-\infty \leq x, y \leq \infty$. So while the expressions look the same, they have subtle differences and we must check that our solutions are, in fact, intersection points.

Thus our intersection points are at $x = -1$, $x = 4$ and $x = 9$. We can check that $\sqrt{|x|} \leq \frac{x}{5} + \frac{6}{5}$ on $[-1, 4]$ and that $\frac{x}{5} + \frac{6}{5} \leq \sqrt{|x|}$ on $[4, 9]$. Therefore we get the integrals:

$$\begin{aligned}
\text{Area} &= \int_{-1}^{4} \left(\frac{x}{5} + \frac{6}{5}\right) - \left(\sqrt{|x|}\right)\, dx + \int_{4}^{9} \left(\sqrt{|x|}\right) - \left(\frac{x}{5} + \frac{6}{5}\right)\, dx \\
&= \int_{-1}^{0} \left[\frac{x}{5} + \frac{6}{5} - \sqrt{-x}\right] dx + \int_{0}^{4} \left[\frac{x}{5} + \frac{6}{5} - \sqrt{x}\right] dx + \int_{4}^{9} \left[\sqrt{x} - \frac{x}{5} - \frac{6}{5}\right] dx \\
&= \left[\frac{x^2}{10} + \frac{6x}{5} + \frac{2(-x)^{3/2}}{3}\right]\Bigg|_{-1}^{0} + \left[\frac{x^2}{10} + \frac{6x}{5} - \frac{2x^{3/2}}{3}\right]\Bigg|_{0}^{4} + \left[\frac{2x^{3/2}}{3} - \frac{x^2}{10} - \frac{6x}{5}\right]\Bigg|_{4}^{9} \\
&= \left[0 - \left(\frac{1}{10} - \frac{6}{5} + \frac{2}{3}\right)\right] + \left[\left(\frac{16}{10} + \frac{24}{5} - \frac{16}{3}\right) - 0\right] + \left[\left(18 - \frac{81}{10} - \frac{54}{5}\right) - \left(\frac{16}{3} - \frac{16}{10} - \frac{24}{5}\right)\right] \\
&= \frac{13}{30} + \frac{16}{15} + \frac{1}{6} \\
&= \frac{5}{3}
\end{aligned}$$

104. $y = 3 - x^2 \quad \text{and} \quad y = -1$

wrt x: We begin by finding the points of intersection:

$$3 - x^2 = -1 \quad \Leftrightarrow \quad 0 = x^2 - 4$$
$$\Leftrightarrow \quad 0 = (x+2)(x-2)$$
$$\Leftrightarrow \quad x = -2 \quad \text{or} \quad x = 2$$

We know $3 - x^2 \geq -1$ on the interval $[-2, 2]$. Therefore we get the following integral:

$$\begin{aligned}
\text{Area} &= \int_{-2}^{2} \left(3 - x^2\right) - (-1)\, dx \\
&= \int_{-2}^{2} 4 - x^2\, dx \\
&= 4x - \frac{1}{3}x^3 \Big|_{-2}^{2} \\
&= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) \\
&= 2\left(8 - \frac{8}{3}\right) \\
&= \frac{32}{3}
\end{aligned}$$

wrt y: To find the area of the region by integrating with respect to y, we note that we want to find the area of the region bounded by $x = \sqrt{3 - y}$ and $x = -\sqrt{3 - y}$ on the interval $[-1, 3]$. (You can get this by looking at the picture of the region.) Therefore, we get the following integral:

$$\begin{aligned}
\text{Area} &= \int_{-1}^{3} \left(\sqrt{3 - y}\right) - \left(-\sqrt{3 - y}\right)\, dy \\
&= 2\int_{-1}^{3} \sqrt{3 - y}\, dy \\
&= 2\left[-\frac{2}{3}(3 - y)^{3/2}\Big|_{-1}^{3}\right] \\
&= 2\left[0 - \left(-\frac{16}{3}\right)\right] \\
&= \frac{32}{3}
\end{aligned}$$

111. Let $F(x)$ be the antiderivative of $f(x) = \frac{\sin x}{x}$ for $x > 0$.

$$\int_{1}^{3} \frac{\sin(2x)}{x}\, dx = \int_{1}^{3} \frac{2\sin(2x)}{2x}\, dx$$

Let $u = 2x, \quad du = 2\,dx$

$$\begin{aligned}\int_1^3 \frac{2\sin(2x)}{2x}\,dx &= \int_2^6 \frac{\sin u}{u}\,du \\ &= \int_2^6 f(u)\,du \\ &= F(6) - F(2) && \text{by FTC (part 2)}\end{aligned}$$

114. a. Suppose f is an odd function

$$\Rightarrow \qquad f(x) = -f(-x)$$

Consider the following:

$$\int_{-a}^{0} f(x)\,dx = \int_{-a}^{0} -f(-x)\,dx$$

If we let $u = -x, \quad du = -dx$, we get the following:

$$\begin{aligned}\int_{-a}^{0} -f(-x)\,dx &= \int_{a}^{0} f(u)\,du \\ &= -\int_{0}^{a} f(u)\,du\end{aligned}$$

If we do a change of variables, we get:

$$\int_{-a}^{0} f(x)\,dx = -\int_{0}^{a} f(x)\,dx$$

Therefore if we look at the integral $\int_{-a}^{a} f(x)\,dx$, we get:

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= 0\end{aligned}$$

b. Consider $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \, dx$

$$
\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \, dx &= -\cos x \,\Big|_{-\pi/2}^{\pi/2} \\
&= -\cos\left(\frac{\pi}{2}\right) - \left(-\cos\left(-\frac{\pi}{2}\right)\right) \\
&= -\cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) && (\textit{cosine is even}) \\
&= 0
\end{aligned}
$$

117. $\int_a^b f(x)\,dx = \int_{a-c}^{b-c} f(x+c)\,dx$

Beginning with the left-hand side, make the substitution $u = x - c, \quad du = dx$. Then we get the following:

$$
\begin{aligned}
\int_a^b f(x)\,dx &= \int_a^b f\left((x-c)+c\right)\,dx \\
&= \int_{a-c}^{b-c} f(u+c)\,du \\
&= \int_{a-c}^{b-c} f(x+c)\,dx && (\textit{change of variables})
\end{aligned}
$$