Math 21B-B - Homework Set 5

Section 6.1:

4. Let's consider the area of a cross section of the solid when $-1 \le x \le 1$. The diameter of the circular cross section at x is given by

$$\text{Diameter} = \left(2 - x^2\right) - x^2 = 2 - 2x^2.$$

Thus, the area of the cross section is given by

$$\text{Area} = \pi \cdot \left[\tfrac{1}{2}\left(2 - 2x^2\right)\right]^2 = \pi \cdot \left(1 - x^2\right)^2 = \pi \cdot \left(1 - 2x^2 + x^4\right).$$

$$\begin{aligned}
\text{Volume} &= \int_{-1}^{1} \pi \cdot \left(1 - 2x^2 + x^4\right)\, dx \\
&= \pi \int_{-1}^{1} \left(1 - 2x^2 + x^4\right)\, dx \\
&= \pi \cdot \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right)\Bigg|_{-1}^{1} \\
&= \pi \cdot \left(1 - \frac{2}{3} + \frac{1}{5}\right) - \pi \cdot \left(-1 + \frac{2}{3} - \frac{1}{5}\right) \\
&= \pi \cdot \left(2 - \frac{4}{3} + \frac{2}{5}\right) \\
&= \frac{16}{15}\pi
\end{aligned}$$

7. a. Let's consider the area of a cross section of the solid when $0 \le x \le \pi$. The base of the triangular cross section at x is given by

$$\text{Base} = 2\sqrt{\sin x}.$$

The height of the triangle is given by

$$\text{Height} = \sqrt{3} \cdot \sqrt{\sin x}. \quad (\textit{Use properties of equilateral triangles})$$

Therefore, the area of the cross section is given by

$$\text{Area} = \left(\tfrac{1}{2}\right)\left(2\sqrt{\sin x}\right)\left(\sqrt{3}\sqrt{\sin x}\right) = \sqrt{3}\sin x.$$

$$
\begin{aligned}
\text{VOLUME} &= \int_0^{\pi} \sqrt{3}\sin x\,dx \\
&= -\sqrt{3}\cos x\Big|_0^{\pi} \\
&= \sqrt{3} - (-\sqrt{3}) \\
&= 2\sqrt{3}
\end{aligned}
$$

b. Let's consider the area of a cross section of the solid when $0 \le x \le \pi$. The base of the square cross section at x is given by

$\text{BASE} = 2\sqrt{\sin x}.$

Therefore, the area of the cross section is given by

$\text{AREA} = (2\sqrt{\sin x})^2 = 4\sin x.$

$$
\begin{aligned}
\text{VOLUME} &= \int_0^{\pi} 4\sin x\,dx \\
&= -4\cos x|_0^{\pi} \\
&= 4 - (-4) \\
&= 8
\end{aligned}
$$

12. a. $\text{DIAMETER} = \frac{2}{\sqrt[4]{1-x^2}}$

$\text{AREA} = \pi \cdot \left[\frac{1}{2}\left(\frac{2}{\sqrt[4]{1-x^2}}\right)\right]^2 = \frac{\pi}{\sqrt{1-x^2}}$

$$
\begin{aligned}
\text{VOLUME} &= \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \frac{\pi}{\sqrt{1-x^2}}\,dx \\
&= \pi \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \frac{1}{\sqrt{1-x^2}}\,dx \\
&= \left(\pi \cdot \sin^{-1} x\right)\Big|_{-\sqrt{2}/2}^{\sqrt{2}/2} \\
&= \pi \cdot \frac{\pi}{4} - \pi \cdot \left(-\frac{\pi}{4}\right) \\
&= 2 \cdot \frac{\pi^2}{4} \\
&= \frac{\pi^2}{2}
\end{aligned}
$$

b. $\text{Diagonal} = \frac{2}{\sqrt[4]{1-x^2}}$

$$\text{Area} = \left(\sqrt{\left(\tfrac{1}{\sqrt[4]{1-x^2}}\right)^2 + \left(\tfrac{1}{\sqrt[4]{1-x^2}}\right)^2}\right)^2 = \tfrac{2}{\sqrt{1-x^2}}$$

$$\begin{aligned}
\text{Volume} &= \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \frac{2}{1-x^2}\,dx \\
&= \left(2\sin^{-1}x\right)\Big|_{-\sqrt{2}/2}^{\sqrt{2}/2} \\
&= 2\cdot\frac{\pi}{4} - 2\cdot\left(-\frac{\pi}{4}\right) \\
&= 4\cdot\frac{\pi}{4} \\
&= \pi
\end{aligned}$$

15. We are rotating about the x-axis, so we want to take the integral with respect to x. We want $0 \le x \le 2$. The radius of a cross section is given by

$\text{Radius} = -\frac{1}{2}x + 1$ (*Between graph and x-axis*)

Therefore, the area of the cross section is given by

$\text{Area} = \pi\cdot\left(-\frac{1}{2}x+1\right)^2 = \pi\cdot\left(\frac{1}{4}x^2 - x + 1\right)$

$$\begin{aligned}
\text{Volume} &= \int_0^2 \pi\cdot\left(\frac{1}{4}x^2 - x + 1\right)\,dx \\
&= \pi\cdot\left(\frac{1}{12}x^3 - \frac{1}{2}x^2 + x\right)\Big|_0^2 \\
&= \pi\cdot\left(\frac{8}{12} - 2 + 2\right) \\
&= \frac{2\pi}{3}
\end{aligned}$$

16. We are rotating about the y-axis, so we want to take the integral with respect to y. We want $0 \le y \le 2$. The radius of the cross section is given by

$\text{Radius} = \frac{3y}{2}$ (*Between the graph and the y-axis*)

Therefore, the area of the cross section is given by

$\text{AREA} = \pi \cdot \left(\frac{3y}{2}\right)^2 = \pi \cdot \frac{9y^2}{4}$

$$\begin{aligned} \text{VOLUME} &= \int_0^2 \pi \cdot \frac{9y^2}{4}\, dy \\ &= \pi \cdot \frac{3y^3}{4}\bigg|_0^2 \\ &= 6\pi \end{aligned}$$

20. We are rotating about the x-axis, so we want to take the integral with respect to y. We want $0 \leq x \leq 2$. The radius of the cross section is given by

    $\text{RADIUS} = x^3$

    Therefore, the area of the cross section is given by

    $\text{AREA} = \pi \left(x^3\right)^2 = \pi \cdot x^6$

$$\begin{aligned} \text{VOLUME} &= \int_0^2 \pi \cdot x^6\, dx \\ &= \frac{\pi}{7} x^7\bigg|_0^2 \\ &= \frac{128\pi}{7} \end{aligned}$$

29. We are rotating about the line $y = \sqrt{2}$, so we want to take the integral with respect to x. We want $0 \leq x \leq \sqrt{2}$. The radius of the cross section is given by

    $\text{RADIUS} = \sqrt{2} - \sec x \tan x \quad$ (*Between* $y = \sqrt{2}$ *and* $y = \sec x \tan x$)

    $\text{AREA} = \pi \cdot (\sqrt{2} - \sec x \tan x)^2 = \pi \cdot \left(2 - 2\sqrt{2} \sec x \tan x + \sec^2 x \tan^2 x\right)$

$$
\begin{aligned}
\textsc{Volume} &= \int_0^{\pi/4} \pi \cdot \left(2 - 2\sqrt{2}\sec x \tan x + \sec^2 x \tan^2 x\right)\, dx \\
&= \pi \cdot \left(2x - 2\sqrt{2}\sec x + \frac{1}{3}\tan^3 x\right)\Bigg|_0^{\pi/4} \qquad (\textit{Use } u\textit{-substitution}) \\
&= \pi \cdot \left(\frac{\pi}{2} - 2\sqrt{2}\cdot\sqrt{2} + \frac{1}{3}\right) - \pi \cdot \left(-2\sqrt{2}\right) \\
&= \pi \cdot \left(\frac{\pi}{2} - 4 + \frac{1}{3} + 2\sqrt{2}\right) \\
&= \pi \cdot \left(\frac{\pi}{2} - \frac{11}{3} + 2\sqrt{2}\right)
\end{aligned}
$$

31. $\textsc{Radius} = \sqrt{5}\, y^2$

$\textsc{Area} = \pi \cdot \left(\sqrt{5}\, y^2\right)^2 = 5\pi y^4$

$$
\begin{aligned}
\textsc{Volume} &= \int_{-1}^{1} 5\pi y^4\, dy \\
&= \pi y^5\Big|_{-1}^{1} \\
&= \pi - (-\pi) \\
&= 2\pi
\end{aligned}
$$

36. $\textsc{Radius} = \frac{\sqrt{2y}}{y^2+1}$

$\textsc{Area} = \pi \cdot \left(\frac{\sqrt{2y}}{y^2+1}\right)^2 = \pi \cdot \frac{2y}{(y^2+1)^2}$

$$
\begin{aligned}
\textsc{Volume} &= \int_0^1 \pi \cdot \frac{2y}{\left(y^2+1\right)^2}\, dy \\
&= \int_1^2 \pi \cdot \frac{1}{u^2}\, du \\
&= \pi \cdot -\frac{1}{u}\Bigg|_1^2 \\
&= -\frac{\pi}{2} + \pi \\
&= \frac{\pi}{2}
\end{aligned}
$$

39. To find the volume of this solid, we are going to use the method of washers. We have outer radius $R(x) = 1$ and inner radius $r(x) = x$.

$$\begin{aligned}
\text{VOLUME} &= \int_0^1 \pi \cdot \left[R(x)^2 - r(x)^2\right]\, dx \\
&= \int_0^1 \pi \cdot \left(1 - x^2\right)\, dx \\
&= \pi \cdot \left(x - \frac{1}{3}x^3\right)\Big|_0^1 \\
&= \pi \cdot \left(1 - \frac{1}{3}\right) - \pi \cdot \left(-1 + \frac{1}{3}\right) \\
&= \frac{4\pi}{3}
\end{aligned}$$

44. We have the outer radius $R(x) = \sec x$ and inner radius $r(x) = \tan x$.

$$\begin{aligned}
\text{VOLUME} &= \int_0^1 \pi \cdot \left(\sec^2 x - \tan^2 x\right)\, dx \\
&= \int_0^1 \pi\, dx \qquad\qquad (\sec^2 x = 1 + \tan^2 x) \\
&= (\pi x)\big|_0^1 \\
&= \pi
\end{aligned}$$

51. a. Rotating about the x-axis

To find the volume of the solid, we will use the method of washers. The outer radius $R = 2$ and the inner radius $r = \sqrt{x}$.

$$\begin{aligned}
\text{VOLUME} &= \int_0^4 \pi \cdot \left(2^2 - \left(\sqrt{x}\right)^2\right)\, dx \\
&= \pi \int_0^4 4 - x\, dx \\
&= \pi \cdot \left(4x - \frac{1}{2}x^2\right)\Big|_0^4 \\
&= \pi \cdot \left(16 - \frac{1}{2} \cdot 16\right) \\
&= 8\pi
\end{aligned}$$

b. Rotating about the y-axis

In this case we will want to integrate with respect to y. Therefore we need to rewrite the the functions with y as the variable. This gives $0 \leq y \leq 2$ and the radius $r = y^2$.

$$\begin{aligned}
\text{Volume} &= \int_0^2 \pi \cdot \left(y^2\right)^2 \, dy \\
&= \pi \int_0^2 y^4 \, dy \\
&= \frac{\pi}{5} y^5 \Big|_0^2 \\
&= \frac{32\pi}{5}
\end{aligned}$$

c. Rotating about the line $y = 2$

In this case, we are integrating with respect to x. We have $0 \leq x \leq 4$. The radius is the distance between the line $y = 2$ and the graph $y = \sqrt{x}$. Therefore, the radius is given by $r = 2 - \sqrt{x}$.

$$\begin{aligned}
\text{Volume} &= \int_0^4 \pi \cdot \left(2 - \sqrt{x}\right)^2 \, dx \\
&= \pi \int_0^4 4 - 4\sqrt{x} + x \, dx \\
&= \pi \cdot \left(4x - \frac{8}{3}x^{3/2} + \frac{1}{2}x^2\right)\Big|_0^4 \\
&= \pi \cdot \left(16 - \frac{8}{3} \cdot 8 + \frac{1}{2} \cdot 16\right) \\
&= \frac{8\pi}{3}
\end{aligned}$$

d. Rotating about the line $x = 4$

In this case we are going to use the washer method, and we will integrate with respect to y. We have $0 \leq y \leq 2$. The outer radius is $R = 4$ and the inner radius is $r = 4 - y^2$.

$$
\begin{aligned}
\textsc{Volume} &= \int_0^2 \pi \cdot \left[4^2 - \left(4 - y^2\right)\right]\, dy \\
&= \pi \int_0^2 16 - \left(16 - 8y^2 + y^4\right)\, dy \\
&= \pi \int_0^2 8y^2 - y^4\, dy \\
&= \pi \cdot \left(\frac{8}{3}y^3 - \frac{1}{5}y^5\right)\Bigg|_0^2 \\
&= \pi \cdot \left(\frac{64}{3} - \frac{32}{5}\right) \\
&= \frac{224\pi}{15}
\end{aligned}
$$

55. We start with the disc given by $x^2 + y^2 \leq a^2$, and we want to revolve it around the line $x = b$ for $b > a$. We will use the method of washers, integrating with respect to y where $-a \leq y \leq a$. The outer radius is $R = b - \left(-\sqrt{a^2 - y^2}\right) = b + \sqrt{a^2 - y^2}$ and the inner radius is given by $r = b - \sqrt{a^2 - y^2}$.

$$
\begin{aligned}
\textsc{Volume} &= \int_{-a}^{a} \pi \cdot \left[\left(b + \sqrt{a^2 - y^2}\right)^2 - \left(b - \sqrt{a^2 - y^2}\right)^2\right]\, dy \\
&= \pi \int_{-a}^{a} \left[\left(b^2 + 2b\sqrt{a^2 - y^2} + \left(a^2 - y^2\right)\right) - \left(b^2 - 2b\sqrt{a^2 - y^2} + \left(a^2 - y^2\right)\right)\right]\, dy \\
&= \pi \int_{-a}^{a} 4b\sqrt{a^2 - y^2}\, dy \\
&= 4b\pi \int_{-a}^{a} \sqrt{a^2 - y^2}\, dy \\
&= 4b\pi \cdot \frac{\pi a^2}{2} \\
&= 2a^2 b\pi^2
\end{aligned}
$$

Section 6.2

1. We are going to integrate with respect to x where $0 \leq x \leq 2$. We have the shell radius $r = x$ and the shell height $h = 1 + \frac{x^2}{4}$.

$$
\begin{aligned}
\text{VOLUME} &= \int_0^2 2\pi r h \, dx \\
&= 2\pi \int_0^2 x\left(1+\frac{x^2}{4}\right) dx \\
&= 2\pi \int_0^2 x + \frac{x^3}{4} \, dx \\
&= 2\pi \cdot \left(\frac{1}{2}x^2 + \frac{x^4}{16}\right)\Bigg|_0^2 \\
&= 2\pi \cdot \left(\frac{1}{2}\cdot 4 + \frac{1}{16}\cdot 16\right) \\
&= 6\pi
\end{aligned}
$$

4. We are going to integrate with respect to y where $0 \le y \le \sqrt{3}$. We have the shell radius $r = y$ and the shell height $h = 3 - \left(3 - y^2\right) = y^2$.

$$
\begin{aligned}
\text{VOLUME} &= \int_0^{\sqrt{3}} 2\pi \cdot y \cdot y^2 \, dy \\
&= 2\pi \int_0^{\sqrt{3}} y^3 \, dy \\
&= 2\pi \cdot \frac{1}{4} y^4 \Bigg|_0^{\sqrt{3}} \\
&= 2\pi \cdot \frac{9}{4} \\
&= \frac{9\pi}{2}
\end{aligned}
$$

8. We are going to integrate with respect to x where $0 \le x \le 1$. We have the shell radius $r = x$ and the shell height $h = 2x - \frac{x}{2} = \frac{3x}{2}$.

$$\begin{aligned}
\textsc{Volume} &= \int_0^1 2\pi x\left(\frac{3x}{2}\right)dx\\
&= 2\pi\int_0^1 \frac{3x^2}{2}\,dx\\
&= 2\pi\cdot\left.\frac{x^3}{2}\right|_0^1\\
&= 2\pi\cdot\frac{1}{2}\\
&= \pi
\end{aligned}$$

13. Let $f(x) = \begin{cases} (\sin x)/x & 0 < x \le \pi \\ 1 & x = 0 \end{cases}$

a. Show that $xf(x) = \sin(x)$ for $0 \le x \le \pi$.

Begin by looking at $xf(x) = \begin{cases} \sin x & 0 < x \le \pi \\ x & x = 0 \end{cases}$.

Notice that $\sin x = x$ when $x = 0$.

Thus $xf(x) = \begin{cases} \sin x & 0 < x \le \pi \\ \sin x & x = 0 \end{cases} = \sin x$

b. We are going to integrate with respect to x where $0 \le x \le \pi$. We have the shell radius $r = x$ and the shell height is given by $h = f(x)$.

$$\begin{aligned}
\textsc{Volume} &= \int_0^\pi 2\pi x f(x)\,dx\\
&= 2\pi\int_0^\pi \sin x\,dx \qquad\qquad (\textit{by part a})\\
&= 2\pi\cdot(-\cos x)|_0^\pi\\
&= 2\pi[-1-(-1)]\\
&= 4\pi
\end{aligned}$$

16. We are going to integrate with respect to y where $0 \le y \le 2$. We have the shell radius $r = y$ and the shell height is given by $h = y^2 - (-y) = y^2 + y$.

$$\begin{aligned}
\textsc{Volume} &= \int_0^2 2\pi \cdot y\left(y^2 + y\right)\, dy \\
&= 2\pi \int_0^2 y^3 + y^2\, dy \\
&= 2\pi \cdot \left(\frac{1}{4}y^4 + \frac{1}{3}y^3\right)\Bigg|_0^2 \\
&= 2\pi \cdot \left(4 + \frac{8}{3}\right) \\
&= \frac{40\pi}{3}
\end{aligned}$$

19. We are going to integrate with respect to y where $0 \leq y \leq 1$. We have the shell radius $r = y$ and the shell height is given by $h = y - (-y) = 2y$.

$$\begin{aligned}
\textsc{Volume} &= \int_0^1 2\pi \cdot (y)(2y)\, dy \\
&= 2\pi \int_0^1 2y^2\, dy \\
&= 2\pi \cdot \frac{2}{3}y^3\Bigg|_0^1 \\
&= 2\pi \cdot \frac{2}{3} \\
&= \frac{4\pi}{3}
\end{aligned}$$

23. In each part, we are going to integrate with respect to y where $0 \leq y \leq 1$. The shell height will always be $h = 12y^2 - 12y^3$, and the shell radius will change depending on which horizontal line we use.

    a. In this case we have the shell radius $r = y$
    (the distance between y and the x-axis).

$$\begin{aligned}
\textsc{Volume} &= \int_0^1 2\pi \cdot y \left(12y^2 - 12y^3\right)\, dy \\
&= 2\pi \int_0^1 12y^3 - 12y^4 \, dy \\
&= 2\pi \cdot \left(3y^4 - \frac{12}{5}y^5\right)\Big|_0^1 \\
&= 2\pi \left(3 - \frac{12}{5}\right) \\
&= \frac{6\pi}{5}
\end{aligned}$$

b. In this case we have the shell radius $r = 1 - y$ (the distance between the line $y = 1$ and y).

$$\begin{aligned}
\textsc{Volume} &= \int_0^1 2\pi(1-y) \left(12y^2 - 12y^3\right)\, dy \\
&= 2\pi \int_0^1 12y^2 - 24y^3 + 12y^4 \, dy \\
&= 2\pi \cdot \left(4y^3 - 6y^4 + \frac{12}{5}y^5\right)\Big|_0^1 \\
&= 2\pi \cdot \left(4 - 6 + \frac{12}{5}\right) \\
&= \frac{4\pi}{5}
\end{aligned}$$

c. In this case we have the shell radius $r = \frac{8}{5} - y$ (the distance between the line $y = \frac{8}{5}$ and y).

$$\begin{aligned}
\textsc{Volume} &= \int_0^1 2\pi \left(\frac{8}{5} - y\right) \left(12y^2 - 12y^3\right)\, dy \\
&= 2\pi \int_0^1 \frac{96}{5}y^2 - \frac{156}{5}y^3 + 12y^4 \, dy \\
&= 2\pi \left(\frac{32}{5}y^3 - \frac{39}{5}y^4 + \frac{12}{5}y^5\right)\Big|_0^1 \\
&= 2\pi \left(\frac{32}{5} - \frac{39}{5} + \frac{12}{5}\right) \\
&= 2\pi
\end{aligned}$$

d. In this case we have the shell radius $r = y - \left(-\frac{2}{5}\right) = y + \frac{2}{5}$ (the distance between y and the line $y = -\frac{2}{5}$)

$$\begin{aligned}
\text{Volume} &= \int_0^1 2\pi\left(y+\frac{2}{5}\right)\left(12y^2 - 12y^3\right)\,dy \\
&= 2\pi\int_0^1 \frac{24}{5}y^2 + \frac{36}{5}y^3 - 12y^4\,dy \\
&= 2\pi\left(\frac{8}{5}y^3 + \frac{9}{5}y^4 - \frac{12}{5}y^5\right)\Big|_0^1 \\
&= 2\pi\left(\frac{8}{5} + \frac{9}{5} - \frac{12}{5}\right) \\
&= 2\pi
\end{aligned}$$

37. Let $W(t) = \int_{f(a)}^{f(t)} \pi\left(\left(f^{-1}(y)\right)^2 - a^2\right)\,dy$ and $S(t) = \int_a^t 2\pi x\left(f(t) - f(x)\right)\,dx$

First, we note that $W(a) = S(a)$.

$$W(a) = \int_{f(a)}^{f(a)} \pi\left(\left(f^{-1}(y)\right)^2 - a^2\right)\,dy = 0 = \int_a^a 2\pi x\left(f(t) - f(x)\right)\,dx = S(a).$$

Second, consider the derivatives of the two functions on the interval $[a, b]$.

$$\begin{aligned}
W'(t) &= \frac{d}{dt}\int_{f(a)}^{f(t)} \pi\left(\left(f^{-1}(y)\right)^2 - a^2\right)\,dy \\
&= \pi\left(\left(f^{-1}\left(f(t)\right)\right)^2 - a^2\right)f'(t) \\
&= \pi\left(t^2 - a^2\right)f'(t)
\end{aligned}$$

$$\begin{aligned}
S'(t) &= \frac{d}{dt}\int_a^t 2\pi x\left(f(t) - f(x)\right)\,dx \\
&= \frac{d}{dt}\int_a^t 2\pi x f(t)\,dx - \frac{d}{dt}\int_a^t 2\pi x f(x)\,dx \\
&= \frac{d}{dt}\left(\pi f(t)x^2\Big|_a^t\right) - 2\pi t f(t) \\
&= \frac{d}{dt}\left(\pi t^2 f(t) - \pi a^2 f(t)\right) - 2\pi t f(t) \\
&= 2\pi t f(t) + \pi t^2 f'(t) - a^2 f'(t) - 2\pi t f(t) \\
&= \pi\left(t^2 - a^2\right)f'(t)
\end{aligned}$$

We know that $W'(t) = S'(t)$ on $[a, b]$ implies that $W(t) = S(t) + C$ on $[a, b]$ where C is some constant. However, we also know that $W(a) = S(a)$ and therefore that $C = 0$. Thus we have that $W(t) = S(t)$ on the interval $[a, b]$.

40. Suppose we try to use the method of shells. We would want to integrate with respect to y where $1 \leq y \leq \sqrt{3}$. The shell radius $r = y$ and the shell height $h = \ln 3 - \ln y^2 = \ln\left(3 - y^2\right)$.

$$\begin{aligned}
\text{Volume} &= \int_1^{\sqrt{3}} 2\pi y \ln\left(3 - y^2\right)\, dy \\
&= \int_2^0 -\pi \ln u \, du \\
&= \int_0^2 \pi \ln u \, du
\end{aligned}$$

Notice, that we do <u>not</u> want to take the integral of $\ln x$, so let's rethink our choice of method.

Suppose we try the method of washers. In this case we will integrate with respect to x where $0 \leq x \leq \ln 3$. The outer radius $R = e^{x/2}$ and the inner radius $r = 1$.

$$\begin{aligned}
\text{Volume} &= \int_0^{\ln 3} \pi \left(\left(e^{x/2}\right)^2 - 1^2\right) dx \\
&= \pi \int_0^{\ln 3} e^x - 1 \, dx \\
&= \pi \left(e^x - x\right)\Big|_0^{\ln 3} \\
&= \pi \left[(3 - \ln 3) - (1 - 0)\right] \\
&= \pi(2 - \ln 3)
\end{aligned}$$

Section 6.3

1. $x = 1 - t, \quad y = 2 + 3t \qquad \text{for } -\frac{2}{3} \leq t \leq 1.$

$$\frac{dx}{dt} = -1 \qquad\qquad \frac{dy}{dt} = 3$$

$$
\begin{aligned}
L &= \int_{-2/3}^{1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_{-2/3}^{1} \sqrt{1+9}\, dt \\
&= \int_{-2/3}^{1} \sqrt{10}\, dt \\
&= \sqrt{10}t\,\Big|_{-2/3}^{1} \\
&= \sqrt{10} - \left(-\frac{2}{3}\right)\sqrt{10} \\
&= \frac{5\sqrt{10}}{3}
\end{aligned}
$$

2. $x = \cos t, \quad y = t + \sin t \qquad \text{for } 0 \le t \le \pi.$

$$
\frac{dx}{dt} = -\sin t \qquad\qquad \frac{dy}{dt} = 1 + \cos t
$$

$$
\begin{aligned}
L &= \int_{0}^{\pi} \sqrt{(-\sin t)^2 + (1+\cos t)^2}\, dt \\
&= \int_{0}^{\pi} \sqrt{\sin^2 t + 1 + 2\cos t + \cos^2 t}\, dt \\
&= \int_{0}^{\pi} \sqrt{2 + 2\cos t}\, dt \\
&= \int_{0}^{\pi} \sqrt{\left(\frac{2-2\cos t}{2-2\cos t}\right)(2+2\cos t)}\, dt \\
&= \int_{0}^{\pi} \sqrt{\frac{4-4\cos^2 t}{2-2\cos t}}\, dt \\
&= \int_{0}^{\pi} \sqrt{\frac{4\sin^2 t}{2-2\cos t}}\, dt \\
&= \int_{0}^{\pi} \frac{2\sin t}{\sqrt{2-2\cos t}}\, dt \\
&= \int_{0}^{4} \frac{1}{\sqrt{u}}\, du \qquad\qquad (u = 2-2\cos t, \quad du = 2\sin t) \\
&= 2\sqrt{u}\Big|_{0}^{4} \\
&= 4
\end{aligned}
$$

4. $x = \frac{t^2}{2}, \quad y = \frac{(2t+1)^{3/2}}{3}$ for $0 \le t \le 4$

$$\frac{dx}{dt} = t \qquad \frac{dy}{dt} = \frac{1}{3} \cdot \frac{3}{2}\sqrt{2t+1} \cdot 2 = \sqrt{2t+1}$$

$$\begin{aligned}
L &= \int_0^4 \sqrt{t^2 + \left(\sqrt{2t+1}\right)^2}\, dt \\
&= \int_0^4 \sqrt{t^2 + 2t + 1}\, dt \\
&= \int_0^4 \sqrt{(t+1)^2}\, dt \\
&= \int_0^4 t + 1\, dt \\
&= \left(\frac{1}{2}t^2 + t\right)\Big|_0^4 \\
&= 8 + 4 \\
&= 12
\end{aligned}$$

7. $x = e^t - t, \quad y = 4e^{t/2}$ for $0 \le t \le 3$.

$$\frac{dx}{dt} = e^t - 1 \qquad \frac{dy}{dt} = 4e^{t/2} \cdot \frac{1}{2} = 2e^{t/2}$$

$$\begin{aligned}
L &= \int_0^3 \sqrt{\left(e^t - 1\right)^2 + \left(2e^{t/2}\right)^2}\, dt \\
&= \int_0^3 \sqrt{e^2t - 2e^t + 1 + 4e^t}\, dt \\
&= \int_0^3 \sqrt{e^2t + 2e^t + 1}\, dt \\
&= \int_0^3 \sqrt{\left(e^t + 1\right)^2}\, dt \\
&= \int_0^3 e^t + 1\, dt \\
&= \left(e^t + t\right)\Big|_0^3 \\
&= \left(e^3 + 3\right) - e^0 \\
&= e^3 + 2
\end{aligned}$$

9. $y = \frac{1}{3}(x^2+2)^{3/2}$ from $x = 0$ to $x = 3$.

$$\frac{dy}{dt} = \frac{1}{3} \cdot \frac{3}{2}(x^2+2)^{1/2} \cdot 2x = x\sqrt{x^2+2}$$

$$\begin{aligned}
L &= \int_0^3 \sqrt{1 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_0^3 \sqrt{1 + \left(x\sqrt{x^2+2}\right)^2}\, dt \\
&= \int_0^3 \sqrt{1 + x^2\left(x^2+2\right)}\, dt \\
&= \int_0^3 \sqrt{x^4 + 2x^2 + 1}\, dt \\
&= \int_0^3 \sqrt{\left(x^2+1\right)^2}\, dt \\
&= \int_0^3 x^2 + 1\, dt \\
&= \left(\frac{1}{3}x^3 + x\right)\Big|_0^3 \\
&= 9 + 3 \\
&= 12
\end{aligned}$$

14. $x = \frac{y^3}{6} + \frac{1}{2y}$ from $y = 2$ to $y = 3$.

$$\frac{dx}{dt} = \frac{y^2}{2} - \frac{1}{2y^2} = \frac{y^4 - 1}{2y^2}$$

$$
\begin{aligned}
L &= \int_2^3 \sqrt{1+\left(\frac{dx}{dt}\right)^2}\,dt \\
&= \int_2^3 \sqrt{1+\left(\frac{y^4-1}{2y^2}\right)^2}\,dt \\
&= \int_2^3 \sqrt{1+\frac{y^8-2y^4+1}{4y^4}}\,dt \\
&= \int_2^3 \sqrt{\frac{y^8+2y^4+1}{4y^4}}\,dt \\
&= \int_2^3 \sqrt{\left(\frac{y^4+1}{2y^2}\right)^2}\,dt \\
&= \int_2^3 \frac{y^4+1}{2y^2}\,dt \\
&= \int_2^3 \frac{y^2}{2}+\frac{1}{2y^2}\,dt \\
&= \left(\frac{y^3}{6}-\frac{1}{2y}\right)\Bigg|_2^3 \\
&= \left(\frac{9}{2}-\frac{1}{6}\right)-\left(\frac{4}{3}-\frac{1}{4}\right) \\
&= \frac{13}{4}
\end{aligned}
$$

18. $x = \displaystyle\int_0^y \sqrt{\sec^4 t - 1}\,dt \quad \text{for} \quad -\tfrac{\pi}{4} \le y \le \tfrac{\pi}{4}.$

$$\frac{dx}{dy} = \sqrt{\sec^4 y - 1}$$

$$
\begin{aligned}
L &= \int_{-\pi/4}^{\pi/4} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy \\
&= \int_{-\pi/4}^{\pi/4} \sqrt{1+\sec^4 y - 1}\, dy \\
&= \int_{-\pi/4}^{\pi/4} \sqrt{\sec^4 y}\, dy \\
&= \int_{-\pi/4}^{\pi/4} \sec^2 y\, dy \\
&= \tan y|_{-\pi/4}^{\pi/4} \\
&= 1-(-1) \\
&= 2
\end{aligned}
$$

33. $x = \ln(\sec t + \tan t) - \sin t, \quad y = \cos t \quad \text{for} \quad 0 \le t \le \frac{\pi}{3}.$

$$
\frac{dx}{dt} = \frac{1}{\sec t + \tan t} \cdot (\sec t \tan t + \sec^2 t) - \cos t = \frac{\sec t(\sec t + \tan t)}{\sec t + \tan t} - \cos t = \sec t - \cos t
$$

$$
\frac{dy}{dt} = -sint
$$

$$
\begin{aligned}
L &= \int_0^{\pi/3} \sqrt{(\sec t - \cos t)^2 + (-\sin t)^2}\, dt \\
&= \int_0^{\pi/3} \sqrt{\sec^2 t - 2 + \cos^2 t + \sin^2 t}\, dt \\
&= \int_0^{\pi/3} \sqrt{\sec^2 t - 1}\, dt \\
&= \int_0^{\pi/3} \sqrt{\tan^2 t}\, dt \\
&= \int_0^{\pi/3} \tan t\, dt \\
&= (\ln|\sec t|)\,|_0^{\pi/3} \\
&= \ln 2
\end{aligned}
$$

35. $x = e^t + e^{-t}, \quad y = 3 - 2t \quad \text{for} \quad 0 \le t \le 3.$

$$
\frac{dx}{dt} = e^t - e^{-t} \qquad\qquad \frac{dy}{dt} = -2
$$

$$
\begin{aligned}
L &= \int_0^3 \sqrt{(e^t - e^{-t})^2 + (-2)^2}\, dt \\
&= \int_0^3 \sqrt{e^2t - 2 + e^{-2t} + 4}\, dt \\
&= \int_0^3 \sqrt{e^2t + 2 + e^{-2t}}\, dt \\
&= \int_0^3 \sqrt{(e^t + e^{-t})^2}\, dt \\
&= \int_0^3 e^t + e^{-t}\, dt \\
&= \left(e^t - e^{-t}\right)\Big|_0^3 \\
&= \left(e^3 - \frac{1}{e^3}\right) - (1 - 1) \\
&= e^3 - \frac{1}{e^3}
\end{aligned}
$$

36. a. $x = \cos(2t), \quad y = \sin(2t) \quad \text{for} \quad 0 \leq t \leq \frac{\pi}{2}$

$$
\frac{dx}{dt} = -2\sin(2t) \qquad\qquad \frac{dy}{dt} = 2\cos(2t)
$$

$$
\begin{aligned}
L &= \int_0^{\pi/2} \sqrt{(-2\sin(2t))^2 + (2\cos(2t))^2}\, dt \\
&= \int_0^{\pi/2} \sqrt{4\sin^2(2t) + 4\cos^2(2t)}\, dt \\
&= \int_0^{\pi/2} \sqrt{4}\, dt \\
&= \int_0^{\pi/2} 2\, dt \\
&= 2t\,\Big|_0^{\pi/2} \\
&= \pi
\end{aligned}
$$

b. $x = \sin(\pi t), \quad y = \cos(\pi t) \quad \text{for} \quad -\frac{1}{2} \leq t \leq \frac{1}{2}$

$$
\frac{dx}{dt} = \pi\cos(\pi t) \qquad\qquad \frac{dy}{dt} = -\pi\sin(\pi t)
$$

$$
\begin{aligned}
L &= \int_{-1/2}^{1/2} \sqrt{(\pi\cos(\pi t))^2 + (-\pi\sin(\pi t))^2}\, dt \\
&= \int_{-1/2}^{1/2} \sqrt{\pi^2\cos^2(\pi t) + \pi^2\sin^2(\pi t)}\, dt \\
&= \int_{-1/2}^{1/2} \sqrt{\pi^2}\, dt \\
&= \int_{-1/2}^{1/2} \pi\, dt \\
&= \pi t\,\Big|_{-1/2}^{1/2} \\
&= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \\
&= \pi
\end{aligned}
$$