Math 21B-B - Homework Set 6

Section 6.5:

1. $y = \tan x, \quad 0 \le x \le \frac{\pi}{4}; \qquad x$-axis

   a.

$$\begin{aligned}\text{Area} &= \int_0^{\pi/4} 2\pi \tan x \sqrt{1 + \left(\frac{d}{dx}\tan x\right)^2}\, dx \\ &= \int_0^{\pi/4} 2\pi \tan x \sqrt{1+\sec^4 x}\, dx\end{aligned}$$

4. $x = \sin y$, hskip .1in $0 \le y \le \pi; \qquad y$-axis

   a.

$$\begin{aligned}\text{Area} &= \int_0^{\pi} 2\pi \sin y \sqrt{1 + \left(\frac{d}{dy}\sin y\right)^2}\, dy \\ &= \int_0^{\pi} 2\pi \sin y \sqrt{1+\cos^2 y}\, dy\end{aligned}$$

9. $y = \frac{x}{2}, \quad 0 \le x \le 4; \qquad x$-axis

Integral Formula:

$$\begin{aligned}\text{Area} &= \int_0^{4} 2\pi \cdot \frac{x}{2} \sqrt{1 + \left(\frac{1}{2}\right)^2}\, dx \\ &= \int_0^4 \frac{\pi\sqrt{5}}{2} x\, dx \\ &= \frac{\pi\sqrt{5}}{4} x^2 \Big|_0^4 \\ &= \frac{\pi\sqrt{5}}{4} \cdot 16 \\ &= \left(4\sqrt{5}\right)\pi\end{aligned}$$

Geometry Formula: Base Circumference $= 2\pi \cdot \frac{4}{2} = 4\pi$
Slant Height $= \sqrt{4^2+2^2} = \sqrt{20} = 2\sqrt{5}$

$$\begin{aligned}\text{Area} &= \frac{1}{2} \cdot 4\pi \cdot 2\sqrt{5} \\ &= \left(4\sqrt{5}\right)\pi\end{aligned}$$

13. $y = \frac{x^3}{9}, \quad 0 \le x \le 2; \qquad x\text{-axis}$

$$\begin{aligned}
\text{Area} &= \int_0^2 2\pi \cdot \frac{x^3}{9}\sqrt{1+\left(\frac{x^2}{3}\right)^2}\,dx \\
&= \frac{2\pi}{9}\int_0^2 x^3\sqrt{1+\frac{x^4}{9}}\,dx \\
&= \frac{2\pi}{9}\int_1^{25/9} \frac{9}{4}\sqrt{u}\,du && u = 1+\frac{x^4}{9}, \quad du = \frac{4}{9}x^3\,dx \\
&= \frac{\pi}{2}\int_1^{25/9} \sqrt{u}\,du \\
&= \frac{\pi}{2}\cdot\frac{2}{3}u^{3/2}\Big|_1^{25/9} \\
&= \frac{\pi}{3}\left[\left(\frac{5}{3}\right)^3 - 1\right] \\
&= \frac{\pi}{3}\left(\frac{125}{27} - 1\right) \\
&= \frac{98\pi}{81}
\end{aligned}$$

14. $y = \sqrt{x}, \quad \frac{3}{4} \le x \le \frac{15}{4}; \qquad x\text{-axis}$

$$\begin{aligned}
\text{Area} &= \int_{3/4}^{15/4} 2\pi\sqrt{x}\cdot\sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\,dx \\
&= 2\pi\int_{3/4}^{15/4} \sqrt{x}\cdot\sqrt{1+\frac{1}{4x}}\,dx \\
&= 2\pi\int_{3/4}^{15/4} \sqrt{x+\frac{1}{4}}\,dx \\
&= 2\pi\int_1^4 \sqrt{u}\,du && u = x+\frac{1}{4}, \quad du = dx \\
&= 2\pi\cdot\frac{2}{3}\,u^{3/2}\Big|_1^4 \\
&= \frac{4\pi}{3}(2^3 - 1^3) \\
&= \frac{28\pi}{3}
\end{aligned}$$

21. $x = \frac{e^y+e^{-y}}{2}, \quad 0 \le y \le \ln 2; \qquad y$-axis

$$\begin{aligned}
\text{Area} &= \int_0^{\ln 2} 2\pi \cdot \frac{e^y+e^{-y}}{2} \cdot \sqrt{1+\left(\frac{e^y-e^{-y}}{2}\right)^2}\,dy \\
&= \pi \int_0^{\ln 2} \left(e^y+e^{-y}\right) \cdot \sqrt{1+\frac{e^{2y}-2+e^{-2y}}{4}}\,dy \\
&= \pi \int_0^{\ln 2} \left(e^y+e^{-y}\right) \cdot \sqrt{\frac{e^{2y}+2+e^{-2y}}{4}}\,dy \\
&= \pi \int_0^{\ln 2} \left(e^y+e^{-y}\right) \cdot \sqrt{\left(\frac{e^y+e^{-y}}{2}\right)^2}\,dy \\
&= \pi \int_0^{\ln 2} \frac{\left(e^y+e^{-y}\right)^2}{2}\,dy \\
&= \frac{\pi}{2} \int_0^{\ln 2} e^{2y}+2+e^{-2y}\,dy \\
&= \frac{\pi}{2} \left(\frac{1}{2}e^{2y}+2y-\frac{1}{2}e^{-2y}\right)\Bigg|_0^{\ln 2} \\
&= \frac{\pi}{2} \left[\left(2+2\ln 2-\frac{1}{8}\right)-\left(\frac{1}{2}-\frac{1}{2}\right)\right] \\
&= \pi \cdot \left(\frac{15}{16}+\ln 2\right)
\end{aligned}$$

24. $y = \cos x, \qquad -\frac{\pi}{2} \le x \le \frac{\pi}{2}; \qquad x$-axis

$$\begin{aligned}
\text{Area} &= \int_{-\pi/2}^{\pi/2} 2\pi \cos x \sqrt{1+(-\sin x)^2}\,dx \\
&= 2\pi \int_{-\pi/2}^{\pi/2} \cos x \sqrt{1+\sin^2 x}\,dx
\end{aligned}$$

28. We want to consider the area of a surface gotten by revolving the arc AB about the x-axis (the area corresponds to the amount of crust for that slice of bread). Therefore we will consider the following area problem:

$$y = \sqrt{r^2-x^2}, \quad a \le x \le a+h; \qquad x\text{-axis}$$

where a and h are numbers such that $0 < h < 2r$ and $-r \le a \le r-h$.

$$\begin{aligned}
\text{AREA} &= \int_a^{a+h} 2\pi\sqrt{r^2-x^2}\cdot\sqrt{1+\left(\frac{d}{dx}\sqrt{r^2-x^2}\right)^2}\,dx \\
&= 2\pi\int_a^{a+h} \sqrt{r^2-x^2}\cdot\sqrt{1+\left(-\frac{x}{\sqrt{r^2-x^2}}\right)^2}\,dx \\
&= 2\pi\int_a^{a+h} \sqrt{r^2-x^2}\cdot\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx \\
&= 2\pi\int_a^{a+h} \sqrt{(r^2-x^2)+x^2}\,dx \\
&= 2\pi\int_a^{a+h} r\,dx \\
&= 2\pi r x\big|_a^{a+h} \\
&= 2\pi\left[(ar+hr)-ar\right] \\
&= 2\pi hr
\end{aligned}$$

We are now down because $\text{AREA} = 2\pi rh$, which is independent of a.

31. a. $y = x, \quad -1 \le x \le 2; \qquad x$-axis

$$\begin{aligned}
\text{AREA} &= \int_{-1}^{2} 2\pi|x|\,ds \\
&= -\int_{-1}^{0} 2\pi x\,ds + \int_0^2 2\pi x\,ds \\
&= -\int_{-1}^{0} 2\pi x\sqrt{1+1}\,dx + \int_0^2 2\pi x\sqrt{1+1}\,dx \\
&= -\sqrt{2}\pi\int_{-1}^{0} 2x\,dx + \sqrt{2}\pi\int_0^2 2x\,dx \\
&= -\sqrt{2}\pi x^2\Big|_{-1}^{0} + \sqrt{2}\pi x^2\Big|_0^2 \\
&= -\sqrt{2}\pi(0-1) + \sqrt{2}\pi(4-0) \\
&= (5\sqrt{2})\pi
\end{aligned}$$

b. $y = \frac{x^3}{9}, \quad -\sqrt{3} \le x \le \sqrt{3}; \qquad x$-axis

$$
\begin{aligned}
\text{Area} &= \int_{-\sqrt{3}}^{\sqrt{3}} \frac{2\pi}{9}\,|x^3|\sqrt{1+\left(\frac{x^2}{3}\right)^2}\,dx \\
&= \frac{2\pi}{9}\int_{-\sqrt{3}}^{\sqrt{3}} |x^3|\sqrt{1+\frac{x^4}{9}}\,dx \\
&= -\frac{2\pi}{9}\int_{-\sqrt{3}}^{0} x^3\sqrt{1+\frac{x^4}{9}}\,dx + \frac{2\pi}{9}\int_{0}^{\sqrt{3}} x^3\sqrt{1+\frac{x^4}{9}}\,dx \\
&= -\frac{2\pi}{9}\int_{2}^{1} \frac{9}{4}\sqrt{u}\,du + \frac{2\pi}{9}\int_{1}^{2} \frac{9}{4}\sqrt{u}\,du \qquad\qquad u = 1+\frac{x^4}{9}, \quad du = \frac{4}{9}x^3\,dx \\
&= \pi\int_{1}^{2} \sqrt{u}\,du \\
&= \frac{2\pi}{3}u^{3/2}\Big|_1^2 \\
&= \frac{2\pi}{3}(2\sqrt{2}-1)
\end{aligned}
$$

If we were to drop the absolute value, we would find that the integral $\int 2\pi f(x)\,ds = 0$. This is because the integrand $\frac{x^3}{9}\cdot\sqrt{1+\frac{x^4}{9}}$ is an odd function and we are integrating over the (symmetric) interval $-\sqrt{3} \le x \le \sqrt{3}$.

33. $x = \cos t, y = 2+\sin t, \quad 0 \le t \le 2\pi, \qquad x$-axis

$$
\begin{aligned}
\text{Area} &= \int_0^{2\pi} 2\pi(2+\sin t)\sqrt{(-\sin t)^2+(\cos t)^2}\,dt \\
&= 2\pi\int_0^{2\pi} (2+\sin t)\sqrt{\sin^2 t+\cos^2 t}\,dt \\
&= 2\pi\int_0^{2\pi} 2+\sin t\,dt \\
&= 2\pi\,(2t-\cos t)\big|_0^{2\pi} \\
&= 2\pi[(4\pi-1)-(0-1)] \\
&= 8\pi^2
\end{aligned}
$$

38. $x = \ln(\sec t + \tan t) - \sin t, y = \cos t, \quad 0 \le t \le \frac{\pi}{3}; \qquad x\text{-axis}$

$$
\begin{aligned}
\text{Area} &= \int_0^{\pi/3} 2\pi \cos t \sqrt{\left(\frac{\sec t \tan t + \sec^2 t}{\sec t + \tan t} - \cos t\right)^2 + (-\sin t)^2}\, dt \\
&= 2\pi \int_0^{\pi/3} \cos t \sqrt{\left(\frac{(\sec t)(\tan t + \sec t)}{\sec t + \tan t} - \cos t\right)^2 + \sin^2 t}\, dt \\
&= 2\pi \int_0^{\pi/3} \cos t \sqrt{(\sec t - \cos t)^2 + \sin^2 t}\, dt \\
&= 2\pi \int_0^{\pi/3} \cos t \sqrt{\sec^2 t - 2 + \cos^2 t + \sin^2 t}\, dt \\
&= 2\pi \int_0^{\pi/3} \cos t \sqrt{\sec^2 t - 1}\, dt \\
&= 2\pi \int_0^{\pi/3} \cos t \sqrt{\tan^2 t}\, dt \\
&= 2\pi \int_0^{\pi/3} \sin t\, dt \\
&= -2\pi \cos t\big|_0^{\pi/3} \\
&= -2\pi \left(\frac{1}{2} - 1\right) \\
&= \pi
\end{aligned}
$$

41. a. We construct the tangent line to our curve f at $m_k = \frac{x_{k-1}+x_k}{2}$. This is the line with slope $f'(m_k)$ that passes through the point $(m_k, f(m_k))$, which is given by the equation.

$$y = f'(m_k)x - f'(m_k)m_k + f(m_k)$$

To find r_1 and r_2, we want to plug x_{k-1} and x_k (respectively) into the equation of the tangent line.

$$
\begin{aligned}
r_1 &= f'(m_k)x_{k-1} - f'(m_k)m_k + f(m_k) \\
&= f'(m_k)(x_{k-1} - m_k) + f(m_k) \\
&= -f'(m_k)\frac{\Delta x_k}{2} + f(m_k)
\end{aligned}
$$

$$
\begin{aligned}
r_2 &= f'(m_k)x_k - f'(m_k)m_k + f(m_k) \\
&= f'(m_k)(x_k - m_k) + f(m_k) \\
&= f'(m_k)\frac{\Delta x_k}{2} + f(m_k)
\end{aligned}
$$

b. By the Pythagorean Theorem, we know

$$\Delta {x_k}^2 + (r_2 - r_1)^2 = {L_k}^2$$

By solving for L_k, we get

$$\begin{aligned} L_k &= \sqrt{(\Delta x_k)^2 + (r_2 - r_1)^2} \\ &= \sqrt{(\Delta x_k)^2 + \left[\left(f(m_k) + f'(m_k)\frac{\Delta x_k}{2}\right) - \left(f(m_k) - f'(m_k)\frac{\Delta x_k}{2}\right)\right]^2} \\ &= \sqrt{(\Delta x_k)^2 + (f'(m_k)\Delta x_k)^2} \end{aligned}$$

c. In order to find the Frustrum Surface Area we need to find y^* (the average height).

$$\begin{aligned} y^* &= \frac{r_1 + r_2}{2} \\ &= \frac{1}{2}\left[\left(f(m_k) - f'(m_k)\frac{\Delta x_k}{2}\right) + \left(f(m_k) + f'(m_k)\frac{\Delta x_k}{2}\right)\right] \\ &= \frac{1}{2} \cdot 2f(m_k) \\ &= f(m_k) \end{aligned}$$

$$\begin{aligned} \text{Frustrum Surface Area} &= 2\pi y^* L \\ &= 2\pi f(m_k)\sqrt{(\Delta x_k)^2 + (f'(m_k)\Delta x_k)^2} \\ &= 2\pi f(m_k)\sqrt{1 + (f'(m_k))^2}\,\Delta x_k \end{aligned}$$

d. If we want to approximate the area of the surface generated by revolving $y = f(x)$ about the x-axis over $[a, b]$, we partition the interval into n pieces and add the frustrum surface areas.

$$\begin{aligned} \text{Area} &\approx \sum_{k=1}^{n} \left(\text{Surface Area of } k^{\text{th}} \text{ frustrum}\right) \\ &= \sum_{k=1}^{n} \left(2\pi f(m_k)\sqrt{1 + (f'(m_k))^2}\,\Delta x_k\right) \end{aligned}$$

To find the actual area of the surface, we will take the limit as $n \to \infty$ of our approximation.

$$\begin{aligned}\text{Area} &= \lim_{n\to\infty}\left[\sum_{k=1}^{n}\left(2\pi f(m_k)\sqrt{1+(f'(m_k))^2}\,\Delta x_k\right)\right]\\ &= \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx\end{aligned}$$

42.

$$\begin{aligned}\text{Area} &= \int_a^b 2\pi f(x)\,dx\\ &= \int_0^{\sqrt{3}} 2\pi\frac{x}{\sqrt{3}}\,dx\\ &= \frac{\pi}{\sqrt{3}}x^2\Big|_0^{\sqrt{3}}\\ &= \frac{\pi}{\sqrt{3}}\cdot 3\\ &= \frac{3\pi}{\sqrt{3}}\\ &= \sqrt{3}\pi\end{aligned}$$

Section 6.6:

4. A force of $90N$ stretches a spring $1m$ beyond its natural length. We can find the spring constant k by using Hooke's Law:

$$90 = k$$

Thus we know that the force it takes to move the spring x meters beyond its natural length is given by $F(x) = 90x$. To find the work it takes to stretch the spring $5m$ beyond its natural length we take the following integral:

$$\begin{aligned}W &= \int_0^5 F(x)\,dx\\ &= \int_0^5 90x\,dx\\ &= 45x^2\Big|_0^5\\ &= 1125J\end{aligned}$$

10. Note that since the force is acting *toward* the origin, $F(x) = -\frac{k}{x^2}$. To find the work done as the particle moves from point b to point a, we can use the following integral:

$$\begin{aligned} W &= \int_a^b -\frac{k}{x^2}\,dx \\ &= \frac{k}{x}\bigg|_a^b \\ &= \frac{k}{b} - \frac{k}{a} \\ &= \frac{k(a-b)}{ab} \end{aligned}$$

11. We know that the general formula for the work done by the piston will look something like $W = \displaystyle\int_{(p_1,V_1)}^{(p_2,V_2)} F(x)\,dx$.

First notice that force is a constant function given by $F = p \cdot A$. Next, if we let x represent the height of the cylinder then the volume of the cylinder is given by $V = Ax$. Thus by looking at the differentials we get that $dV = A\,dx$. Substituting this information into our original integral equation for work we get:

$$\begin{aligned} W &= \int_{(p_1,V_1)}^{(p_2,V_2)} (p \cdot A)\left(\frac{dV}{A}\right) \\ &= \int_{(p_1,V_1)}^{(p_2,V_2)} p\,dV \end{aligned}$$

12. We want to find the work done in compressing the gas from $V_1 = 243\,\text{in.}^3$ to $V_2 = 32\,\text{in.}^3$, where $p_1 = 50\,\text{lb/in.}^3$.

We assume that the p and V obey the gas law, which states that $pV^{1.4} = c$. We can solve for c using the fact that $p_1 = 50$ and $V_1 = 243$:

$$c = 50 \cdot 243^{1.4} = 50 \cdot 2187 = 109350$$

Thus we have $pV^{1.4} = 109350$. If we solve for p in terms of V, we get that $p = 109350V^{-1.4}$. Using the integral from (11), we find that the work done to compress the gas from (p_1, V_1) to (p_2, V_2) is given by:

$$
\begin{aligned}
W &= \int_{(p_1,V_1)}^{(p_2,V_2)} p\,dV \\
&= \int_{243}^{32} 109350V^{-1.4}\,dV \\
&= -\int_{32}^{243} 109350V^{-1.4}\,dV \\
&= \left. 273375V^{-0.4}\right|_{32}^{243} \\
&= 273375\left(\frac{1}{9}-\frac{1}{4}\right) \\
&= 273375\cdot -\frac{5}{36} \\
&= -37968.75 \quad \text{in}\cdot\text{lb}
\end{aligned}
$$

15. a. Work to empty the tank by pumping the water back to ground level.

Consider a horizontal "slab" of water at level y with width Δy. The force F_{slab} to lift the slab is given by:

$$
\begin{aligned}
F_{slab} &= 62.4\cdot V_{slab} \\
&= 62.4\cdot \Delta y\cdot 10\cdot 12 \\
&= 62.4\cdot 120\,\Delta y\,lb
\end{aligned}
$$

To compute the work needed to pump this slab out of the tank, we recall that F_{slab} must act over a distance of $y$ft. Thus we have:

$$W_{slab} = F_{slab}\cdot d = 62.4\cdot 120y\Delta y\,ft\cdot lb$$

To approximate the total work W necessary to empty the tank, we could use a Riemann sum $f(y) = 7488y$ over the interval $0 \le y \le 20$.

$$W = \sum_{0}^{20} 64.2\cdot 120y\Delta y\,ft\cdot lb$$

To find the exact value, we take the limit of the this sum over progressively finer partitions.

$$
\begin{aligned}
W &= \int_0^{20} 64.2 \cdot 120y\, dy \\
&= \int_0^{20} 7488y\, dy \\
&= 3744y^2\Big|_0^{20} \\
&= 3744 \cdot 400 \\
&= 1497600\, ft \cdot lb
\end{aligned}
$$

b. The pump moves 250 $ft - lb/sec$, the time it will take to empty the tank is:

$$\text{time} = \tfrac{1497600\, ft \cdot lb}{250\, ft - lb/sec} = 5990.4\, sec \approx 1\text{hr } 40\text{min}$$

c. The amount of work it takes to empty out the first half of the tank is given by:

$$
\begin{aligned}
\text{Work} &= \int_0^{10} 7488y\, dy \\
&= 3744y^2\Big|_0^{10} \\
&= 3744 \cdot 100 \\
&= 374400\, \text{ft-lb}
\end{aligned}
$$

To see how much time this amount of work will take we use:

$$t = \tfrac{374400}{250} = 1497.6\, \text{sec}$$

The last thing to note is that $1497.6\,\text{sec} \approx 25\,\text{min}$.

d. i. If water weighs $62.26\,\text{lb/ft}^3$

$$\begin{aligned} W &= \int_0^{20} 62.26 \cdot 120y\,dy \\ &= \int_0^{20} 7471.2y\,dy \\ &= 3735.2y^2\Big|_0^{20} \\ &= 3735.6 \cdot 400 \\ &= 1494240\,\text{ft-lb} \end{aligned}$$

$$\begin{aligned} t &= \frac{1494240}{250} \\ &= 5976.96\,\text{sec} \\ &\approx 1\text{hr}\ 40\text{min} \end{aligned}$$

ii. If water weighs $62.59\,\text{lb/ft}^3$

$$\begin{aligned} W &= \int_0^{20} 62.59 \cdot 120y\,dy \\ &= \int_0^{20} 7510.8y\,dy \\ &= 3755.4y^2\Big|_0^{20} \\ &= 3755.4 \cdot 400 \\ &= 1502160\,\text{ft-lb} \end{aligned}$$

$$\begin{aligned} t &= \frac{1502160}{250} \\ &= 6008.64\,\text{sec} \\ &\approx 1\text{hr}\ 40\text{min} \end{aligned}$$

25. We know to begin with that $W = \displaystyle\int_{x_1}^{x_2} F(x)\,dx$. Using Newton's second

law, this gives us:

$$\begin{aligned} W &= \int_{x_1}^{x_2} m\nu \cdot \frac{d\nu}{dx}\,dx \\ &= m \int_{x_1}^{x_2} \left(\nu \cdot \frac{d\nu}{dx}\right)\,dx \\ &= m \int_{\nu_1}^{\nu_2} u\,du \\ &= \frac{m}{2} u^2 \Big|_{\nu_1}^{\nu_2} \\ &= \frac{1}{2} m{\nu_2}^2 - \frac{1}{2} m{\nu_1}^2 \end{aligned}$$

37.

$$\begin{aligned} W &= \int_{6370000}^{35780000} \frac{1000MG}{r^2}\,dr \\ &= -\frac{1000MG}{r}\Big|_{6370000}^{35780000} \\ &= \left(5.975 \times 10^{24}\right)\left(6.6720 \times 10^{-11}\right)\left(-\frac{1}{3578} + \frac{1}{637}\right) \\ &\approx 5.144 \times 10^{10} J \end{aligned}$$

38. a. Let ρ be the x-coordinate of the second electron. Then $r^2 = (\rho - 1)^2$.

$$\begin{aligned} W &= \int_{-1}^{0} F(r)\,dr \\ &= \int_{-1}^{0} \frac{23 \times 10^{-29}}{(\rho - 1)^2}\,d\rho \\ &= \left(23 \times 10^{-29}\right) \cdot -\frac{1}{\rho - 1}\Big|_{-1}^{0} \\ &= \frac{1}{2}\left(23 \times 10^{-29}\right) \\ &= 11.5 \times 10^{-29} \end{aligned}$$

b. We will use the fact that $W = W_1 + W_2$ where W_1 is the work against the fixed electron $(-1, 0)$ and W_2 is the work against the second fixed electron $(1, 0)$. We will let ρ be the x-coordinate of the third electron. Then ${r_1}^2 = (\rho + 1)^2$ and ${r_2}^2 = (\rho - 1)^2$.

$$
\begin{aligned}
W_1 &= \int_3^5 \frac{23 \times 10^{-29}}{(\rho+1)^2}\, d\rho \\
&= -\frac{23 \times 10^{-29}}{\rho+1}\bigg|_3^5 \\
&= -(23 \times 10^{-29}) \left(\frac{1}{6} - \frac{1}{4}\right) \\
&= \frac{23}{12} \times 10^{-29}
\end{aligned}
$$

$$
\begin{aligned}
W_2 &= \int_3^5 \frac{23 \times 10^{-29}}{(\rho-1)^2}\, d\rho \\
&= -\frac{23 \times 10^{-29}}{\rho-1}\bigg|_3^5 \\
&= -(23 \times 10^{-29}) \left(\frac{1}{4} - \frac{1}{2}\right) \\
&= \frac{23}{4} \times 10^{-29}
\end{aligned}
$$

$$
\begin{aligned}
W &= W_1 + W_2 \\
&= \frac{23}{12} \times 10^{-29} + \frac{23}{4} \times 10^{-29} \\
&= \frac{23}{3} \times 10^{-29}
\end{aligned}
$$