Math 21B-B - Homework Set 7

Section 6.7:

1. Calculate the fluid force on one side of the triangular plate:

   Note that in this coordinate system, we have $-5 \leq y \leq -2$. We have that STRIP DEPTH $= -y$ and $L(y) = 2(y+5)$ (using the fact that the right leg of the triangle lies on the line $y = x - 5$ and the fact that our triangle is symmetric over the y-axis).

$$\begin{aligned}
\text{Fluid Force} &= \int_a^b w \cdot (\text{strip depth}) \cdot L(y)\, dy \\
&= \int_{-5}^{-2} 62.4 \cdot -y \cdot (2y+10)\, dy \\
&= \int_{-5}^{-2} -124.8y^2 - 624y\, dy \\
&= -41.6y^3 - 312y^2 \Big|_{-5}^{-2} \\
&= [(-41.6 \cdot -8) - (312 \cdot 4)] - [(-41.6 \cdot -125) - (312 \cdot 25)] \\
&= 1684.8\,\text{lb}
\end{aligned}$$

11. a. What is the fluid force on the gate when the liquid is 2 ft deep?

   We will assume that the vertex of the gate is along the bottom edge of the cubical tank. Using the coordinates given in the illustration in the book, we know that the water level is at the line $y = 2$. To calculate the force we are concerned with $0 \leq y \leq 1$. We have that the STRIP DEPTH $= 2 - y$ and $L(y) = 2\sqrt{y}$ (using the symmetry of the gate over the y-axis).

$$\begin{aligned}
\text{Fluid Force} &= \int_0^1 50 \cdot (2-y) \cdot 2\sqrt{y}\, dy \\
&= \int_0^1 200\sqrt{y} - 100y^{3/2}\, dy \\
&= \frac{400}{3} y^{3/2} - 40y^{5/2} \Big|_0^1 \\
&= \frac{400}{3} - 40 \\
&= \frac{280}{3} \\
&\approx 93.33\,\text{lb}
\end{aligned}$$

b. What is the maximum height to which the container can be filled without exceeding its design limitation? Suppose that the water level is at $y = h$ for some positive number h. Then to calculate the force on the gate we are still just concerned with $0 \leq y \leq 1$ and we have that STRIP DEPTH $= h - y$ and $L(y) = 2\sqrt{y}$. Thus treating h as a number we can calculate the force on the gate in terms of h by using our integral formula:

$$\begin{aligned}\text{Fluid Force} &= \int_0^1 50 \cdot (h - y) \cdot 2\sqrt{y}\,dy \\ &= \int_0^1 100h\sqrt{y} - 100y^{3/2}\,dy \\ &= \frac{200h}{3}y^{3/2} - 40y^{5/2}\Big|_0^1 \\ &= \frac{200h}{3} - 40\end{aligned}$$

We know that the gate is designed to withstand FLUID FORCE ≤ 160 lb. We substitute our above result for the force into this inequality to find the maximum height to which the container can be filled.

$$\begin{aligned}\text{Fluid Force} &\leq 160 \\ \frac{200h}{3} - 40 &\leq 160 \\ \frac{200h}{3} &\leq 200 \\ h &\leq 3\end{aligned}$$

Therefore, we see that the maximum height is $h = 3$ft.

17. Consider the amount of force on the end when the tank is completely full. In that case we have that $-2 \leq y \leq 0$, where STRIP DEPTH $= -y$ and $L(y) = 2\sqrt{4 - y^2}$ (by the symmetry of the semicircular ends).

$$\begin{aligned}
\text{Fluid Force} &= \int_{-2}^{0} 62.4 \cdot (-y) \cdot 2\sqrt{4-y^2}\, dy \\
&= \int_{-2}^{0} 62.4(-2y)\sqrt{4-y^2}\, dy \\
&= \int_{0}^{4} 62.4\sqrt{u}\, du \qquad\qquad (u = 4 - y^2, \quad du = -2y\, dy) \\
&= 41.6\, u^{3/2}\Big|_0^4 \\
&= 41.6 \cdot 8 \\
&= 332.8\,\text{lb}
\end{aligned}$$

If we plug this force into Hooke's Law we can see how far we will compress the spring from its resting state:

$$\begin{aligned}
\text{Force} &= (\text{spring constant}) \cdot (\text{distance}) \\
332.8 &= 100x \\
x &\approx 3.33\,\text{ft}
\end{aligned}$$

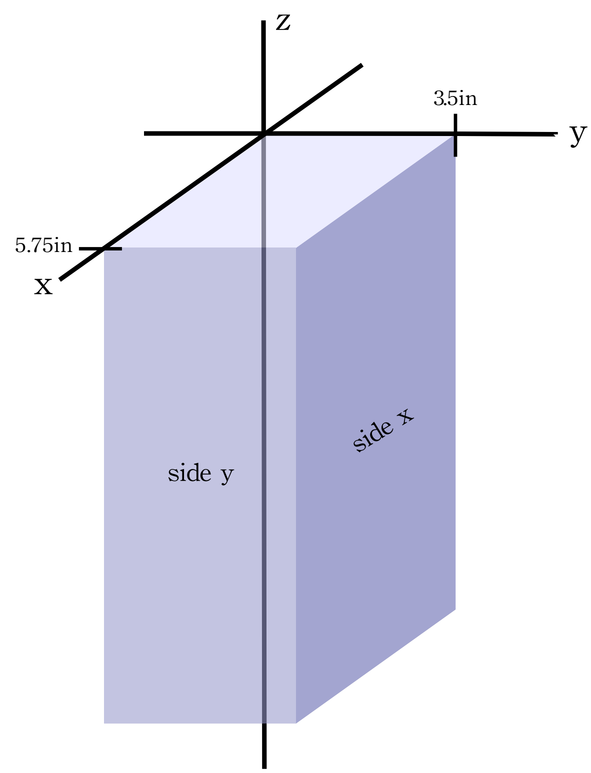
This means that the spring will only be compressed 3.33 ft $<$ 5 ft when the tank is full. Therefore, we know that the tank will overflow.

20. Recall that the weight density of olive oil is 57 lb/ft^3. In specific, note that the units involve feet, so we will need to convert the sidelengths of the can into feet. Thus we have that the base of the can is $\frac{23}{48}$ ft $\times$ $\frac{7}{24}$ ft and the base is $\frac{5}{6}$ ft.

    If the can is full, we can assume that the base is at a constant depth. Therefore, we can compute the force on the base by the formula $F = whA$.

$$\begin{aligned}
F_{base} &= 57 \cdot \frac{5}{6} \cdot \frac{23}{48} \cdot \frac{7}{24} \\
&= \frac{45885}{6912} \\
&\approx 6.64\,\text{lb}
\end{aligned}$$

Now to compute the force on the two pairs of sides, we will need to use our integral formula because the depth varies along the sides. Let F_x denote the force on the pair of $\frac{23}{48}$ ft $\times \frac{5}{6}$ ft sides and F_y denote the force on the pair of $\frac{7}{24}$ ft $\times \frac{5}{6}$ ft sides.



$$\begin{aligned} F_x &= \int_{-5/6}^{0} 57 \cdot (-y) \cdot \frac{23}{48}\, dy \\ &= \int_{-5/6}^{0} -\frac{1311}{48} y\, dy \\ &= -\frac{1311}{96} y^2 \Big|_{-5/6}^{0} \\ &= \frac{1311}{96} \cdot \left(-\frac{5}{6}\right)^2 \\ &= \frac{32775}{3456} \\ &\approx 9.48\,\text{lb} \end{aligned}$$

$$\begin{aligned} F_y &= \int_{-5/6}^{0} 57 \cdot (-y) \frac{7}{24}\, dy \\ &= \int_{-5/6}^{0} -\frac{399}{24} y\, dy \\ &= -\frac{399}{48} y^2 \Big|_{-5/6}^{0} \\ &= \frac{399}{48} \cdot \left(-\frac{5}{6}\right)^2 \\ &= \frac{9975}{1728} \\ &\approx 5.77\,\text{lb} \end{aligned}$$

Section 7.1:

3.

$$\begin{aligned} \int \frac{2y}{y^2 - 25}\, dy &= \int \frac{1}{u}\, du \qquad (u = y^2 - 25, \quad du = 2y\, dy) \\ &= \ln|u| + C \\ &= \ln|y^2 - 25| + C \end{aligned}$$

6.

$$\int \frac{\sec y \tan y}{2+\sec y}\,dy = \int \frac{1}{u}\,du \qquad (u = 2+\sec y, \quad du = \sec y \tan y\,dy)$$
$$= \ln|u| + C$$
$$= \ln|2+\sec y| + C$$

8.

$$\int \frac{\sec x}{\sqrt{\ln(\sec x + \tan x)}}\,dx = \int \frac{1}{\sqrt{u}}\,du \qquad \left(u = \ln(\sec x + \tan x),\ du = \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}\,dx = \sec x\,dx\right)$$
$$= 2\sqrt{u} + C$$
$$= 2\sqrt{\ln(\sec x + \tan x)} + C$$

15.

$$\int \frac{e^{\sqrt{r}}}{\sqrt{r}}\,dr = \int 2e^u\,du \qquad \left(u = \sqrt{r}, \quad du = \frac{1}{2\sqrt{r}}\,dr\right)$$
$$= 2e^u + C$$
$$= 2e^{\sqrt{r}} + C$$

20.

$$\int \frac{e^{-1/x^2}}{x^3}\,dx = \int \frac{1}{2}\,e^u\,du \qquad \left(u = -\frac{1}{x^2}, \quad du = \frac{2}{x^3}\,dx\right)$$
$$= \frac{1}{2}\,e^u + C$$
$$= \frac{1}{2}\,e^{-1/x^2} + C$$

51. $\dfrac{dy}{dx} = 1 + \dfrac{1}{x}, \quad y(1) = 3$

$$y = \int 1 + \frac{1}{x} \qquad \left(y = \int \frac{dy}{dx}\,dx\right)$$
$$= x + \ln|x| + C$$

We will use the initial condition $y(1) = 3$ to find the value of C.

$$y(1) = 3 \quad \Rightarrow \quad 1 + \ln 1 + C = 3$$
$$\Rightarrow \quad C = 2$$

Thus we get $y = x + \ln|x| + 2$.

52. $\dfrac{d^2y}{dx^2} = \sec^2 x, \quad y(0) = 0 \quad y'(0) = 1$

$$\frac{dy}{dx} = \int \sec^2 x\,dx \qquad \left(\frac{dy}{dx} = \int \frac{d^2y}{dx^2}\,dx\right)$$
$$= \tan x + C_1$$

We will use the initial condition $y'(0) = 1$ to find C_1.

$$\begin{aligned} y'(0) = 1 \quad &\Rightarrow \quad \tan(0) + C_1 = 1 \\ &\Rightarrow \quad C_1 = 1 \end{aligned}$$

Thus we get $\dfrac{dy}{dx} = \tan x + 1$. We now integrate $\dfrac{dy}{dx}$ to find y.

$$\begin{aligned} y &= \int \tan x + 1\,dx \qquad \left(y = \int \frac{dy}{dx}\,dx\right) \\ &= \int \frac{\sin x}{\cos x} + 1\,dx \\ &= \int -\frac{1}{u} + 1\,du \qquad (u = \cos x, \quad du = -\sin x\,dx) \\ &= -\ln|u| + u + C_2 \\ &= -\ln|\cos x| + \cos x + C_2 \\ &= \ln|\sec x| + \cos x + C_2 \end{aligned}$$

We will use the initial condition $y(0) = 0$ to find C_2.

$$\begin{aligned} y(0) = 0 \quad &\Rightarrow \quad \ln|\sec(0)| + \cos(0) + C_2 = 0 \\ &\Rightarrow \quad \ln(1) + 1 + C_2 = 0 \\ &\Rightarrow \quad 1 + C_2 = 0 \\ &\Rightarrow \quad C_2 = -1 \end{aligned}$$

Thus we get $y = \ln|\sec x| + \cos x - 1$.

57. a.

$$\begin{aligned} L(x) &= f(0) + f'(0) \cdot x \\ &= \ln(1+0) + \frac{1}{0+1} \cdot x \\ &= x \end{aligned}$$

b. Let $f(x) = \ln(1+x)$ and consider $f'(x)$ and $f''(x)$.

$$f'(x) = \tfrac{1}{1+x} \qquad\qquad f''(x) = -\tfrac{1}{(1+x)^2}$$

On the interval $[0, 0.1]$ we see that $f(x)$ is increasing (by $f'(x)$) and concave down (by $f''(x)$). Using what we know about the graphs of $f(x)$ and $L(x)$, we can see that the maximum approximation error on $[0, 0.1]$ occurs at $x = 0.1$.

Therefore, the maximum error $= L(0.1) - f(0.1) \approx 0.00469$.

58. a.

$$\begin{aligned} L(x) &= f(0) + f'(0) \cdot x \\ &= e^0 + e^0 \cdot x \\ &= 1 + x \end{aligned}$$

Section 7.2:

2. a. $\dfrac{dp}{dh} = kp \quad (k\,\text{constant}) \qquad\qquad p = p_0$ when $h = 0$

Using the Law of Exponential Change (p509) we know that $p = p_0\, e^{kh}$. In the problem we are given that $p(0) = 1013$ and $p(20) = 90$. We will use these initial conditions to find the values of p_0 and k.

$$\begin{aligned} p(0) = 1013 \quad &\Rightarrow \quad p_0\, e^0 = 1013 \\ &\Rightarrow \quad p_0 = 1013 \end{aligned}$$

Thus, $p(h) = 1013e^{kh}$

$$\begin{aligned} p(20) = 90 \quad &\Rightarrow \quad 1013\, e^{20k} = 90 \\ &\Rightarrow \quad e^{20k} = \frac{90}{1013} \\ &\Rightarrow \quad 20k = \ln\left(\frac{90}{1013}\right) \\ &\Rightarrow \quad k = \frac{1}{20}\ln\left(\frac{90}{1013}\right) \approx -0.121 \end{aligned}$$

Thus, $p(h) \approx 1013e^{-0.121h}$

b. $p(50) = 1013\, e^{(-0.121)(50)} \approx 2.389$ milibars

c. $900 = 1013\, e^{-0.121h}$

$$\Leftrightarrow \tfrac{900}{1013} = e^{-0.121h}$$

$$\Leftrightarrow \ln\left(\tfrac{900}{1013}\right) = -0.121h$$

$$\Leftrightarrow h \approx 0.977 \text{ km}$$

6. $\frac{dV}{dt} = -\frac{1}{40} V \qquad \Rightarrow \qquad V = V_0\, e^{-t/40}$

   We want to find t such that $V(t) = 0.1V_0$.

$$\begin{aligned} 0.1V_0 = V_0\, e^{-t/40} \qquad &\Rightarrow \qquad 0.1 = e^{-t/40} \\ &\Rightarrow \qquad \ln(0.1) = -\frac{t}{40} \\ &\Rightarrow \qquad t = -40\ln(0.1) \approx 92.1 \text{ sec} \end{aligned}$$

7. We will let the population of the bacteria colony be given by $p(t) = e^{kt}$ where t in measured in hours (we are told $p_0 = 1$).

   We know that the p doubles every half hour. Thus $p(0.5) = 2$. We can use this information to find the value of k.

$$\begin{aligned} 2 = e^{0.5k} \qquad &\Rightarrow \qquad \ln(2) = 0.5k \\ &\Rightarrow \qquad k = 2\ln(2) \\ &\Rightarrow \qquad k = \ln(4) \end{aligned}$$

   Thus we have that $p(t) = e^{\ln(4)t}$. In 24 hours there are $p(24) = e^{\ln(4)\cdot 24} = 4^{24} \approx 2.81475 \times 10^{14}$ bacteria.

21. We know that the temperature (as a function of time) is given by:

$$H(t) = 20 + (90 - 20)\, e^{-kt} = 20 + 70e^{-kt}$$

   .

   We know that $H(10) = 60$. Using this information, we can solve for the constant k.

$$\begin{aligned} 20 + 70e^{-10k} = 60 \qquad &\Rightarrow \qquad 70e^{-10k} = 40 \\ &\Rightarrow \qquad e^{-10k} = \frac{4}{7} \\ &\Rightarrow \qquad -10k = \ln\left(\frac{4}{7}\right) \\ &\Rightarrow \qquad k = -\frac{1}{10}\ln\left(\frac{4}{7}\right) \\ &\Rightarrow \qquad k \approx 0.05596 \end{aligned}$$

Thus, we have $H(t) = 20 + 70e^{-0.05596t}$.

a.

$$\begin{aligned} 20 + 70e^{-0.05596t} = 35 \quad &\Rightarrow \quad 70e^{-0.05596t} = 15 \\ &\Rightarrow \quad e^{-0.05596t} = \frac{3}{14} \\ &\Rightarrow \quad -0.05596t = \ln\left(\frac{3}{14}\right) \\ &\Rightarrow \quad t \approx 27.5 \text{ min} \end{aligned}$$

We know that it took 10 minutes for the temperature to reach 60°C, so it takes an additional 17.5 minutes for the temperature to reach 35°C.

b. Now we are changing H_S to -15°C.

$H(t) = -15 + (90 + 15)e^{-0.05596t} = -15 + 105e^{-0.05596t}$

$$\begin{aligned} -15 + 105e^{-0.05596t} = 35 \quad &\Rightarrow \quad 105e^{-0.05596t} = 50 \\ &\Rightarrow \quad e^{-0.05596} = \frac{10}{21} \\ &\Rightarrow \quad -0.05596t = \ln\left(\frac{10}{21}\right) \\ &\Rightarrow \quad t \approx 13.26 \text{ min} \end{aligned}$$