Math 21B-B - Homework Set 8

Section 7.1:

3.

$$\begin{aligned}\int \frac{2y}{y^2-25}\,dy &= \int \frac{1}{u}\,du \qquad\qquad (u = y^2-25, \quad du = 2y\,dy)\\ &= \ln|u| + C\\ &= \ln|y^2-25| + C\end{aligned}$$

6.

$$\begin{aligned}\int \frac{\sec y \tan y}{2+\sec y}\,dy &= \int \frac{1}{u}\,du \qquad\qquad (u = 2+\sec y, \quad du = \sec y \tan y\,dy)\\ &= \ln|u| + C\\ &= \ln|2+\sec y| + C\end{aligned}$$

8.

$$\begin{aligned}\int \frac{\sec x}{\sqrt{\ln(\sec x + \tan x)}}\,dx &= \int \frac{1}{\sqrt{u}}\,du \qquad (u = \ln(\sec x + \tan x),\ du = \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}\,dx = \sec x\,dx)\\ &= 2\sqrt{u} + C\\ &= 2\sqrt{\ln(\sec x + \tan x)} + C\end{aligned}$$

15.

$$\begin{aligned}\int \frac{e^{\sqrt{r}}}{\sqrt{r}}\,dr &= \int 2e^u\,du \qquad\qquad (u = \sqrt{r}, \quad du = \frac{1}{2\sqrt{r}}\,dr)\\ &= 2e^u + C\\ &= 2e^{\sqrt{r}} + C\end{aligned}$$

20.

$$\begin{aligned}\int \frac{e^{-1/x^2}}{x^3}\,dx &= \int \frac{1}{2}\,e^u\,du \qquad\qquad (u = -\frac{1}{x^2}, \quad du = \frac{2}{x^3}\,dx)\\ &= \frac{1}{2}\,e^u + C\\ &= \frac{1}{2}\,e^{-1/x^2} + C\end{aligned}$$

51. $\frac{dy}{dx} = 1 + \frac{1}{x}, \quad y(1) = 3$

$$\begin{aligned} y &= \int 1 + \frac{1}{x} && \left(y = \int \frac{dy}{dx}\,dx\right) \\ &= x + \ln|x| + C \end{aligned}$$

We will use the initial condition $y(1) = 3$ to find the value of C.

$$\begin{aligned} y(1) = 3 \quad &\Rightarrow \quad 1 + \ln 1 + C = 3 \\ &\Rightarrow \quad C = 2 \end{aligned}$$

Thus we get $y = x + \ln|x| + 2$.

52. $\frac{d^2y}{dx^2} = \sec^2 x, \quad y(0) = 0 \quad y'(0) = 1$

$$\begin{aligned} \frac{dy}{dx} &= \int \sec^2 x\,dx && \left(\frac{dy}{dx} = \int \frac{d^2y}{dx^2}\,dx\right) \\ &= \tan x + C_1 \end{aligned}$$

We will use the initial condition $y'(0) = 1$ to find C_1.

$$\begin{aligned} y'(0) = 1 \quad &\Rightarrow \quad \tan(0) + C_1 = 1 \\ &\Rightarrow \quad C_1 = 1 \end{aligned}$$

Thus we get $\frac{dy}{dx} = \tan x + 1$. We now integrate $\frac{dy}{dx}$ to find y.

$$\begin{aligned} y &= \int \tan x + 1\,dx && \left(y = \int \frac{dy}{dx}\,dx\right) \\ &= \int \frac{\sin x}{\cos x} + 1\,dx \\ &= \int -\frac{1}{u} + 1\,du && (u = \cos x, \quad du = -\sin x\,dx) \\ &= -\ln|u| + u + C_2 \\ &= -\ln|\cos x| + \cos x + C_2 \\ &= \ln|\sec x| + \cos x + C_2 \end{aligned}$$

We will use the initial condition $y(0) = 0$ to find C_2.

$$\begin{aligned} y(0) = 0 \quad &\Rightarrow \quad \ln|\sec(0)| + \cos(0) + C_2 = 0 \\ &\Rightarrow \quad \ln(1) + 1 + C_2 = 0 \\ &\Rightarrow \quad 1 + C_2 = 0 \\ &\Rightarrow \quad C_2 = -1 \end{aligned}$$

Thus we get $y = \ln|\sec x| + \cos x - 1$.

57. a.

$$\begin{aligned} L(x) &= f(0) + f'(0) \cdot x \\ &= \ln(1+0) + \frac{1}{0+1} \cdot x \\ &= x \end{aligned}$$

b. Let $f(x) = \ln(1+x)$ and consider $f'(x)$ and $f''(x)$.

$f'(x) = \frac{1}{1+x}$ $\qquad$ $f''(x) = -\frac{1}{(1+x)^2}$

On the interval $[0, 0.1]$ we see that $f(x)$ is increasing (by $f'(x)$) and concave down (by $f''(x)$). Using what we know about the graphs of $f(x)$ and $L(x)$, we can see that the maximum approximation error on $[0, 0.1]$ occurs at $x = 0.1$.

Therefore, the maximum error $= L(0.1) - f(0.1) \approx 0.00469$.

58. a.

$$\begin{aligned} L(x) &= f(0) + f'(0) \cdot x \\ &= e^0 + e^0 \cdot x \\ &= 1 + x \end{aligned}$$

Section 7.2:

2. a. $\dfrac{dp}{dh} = kp \quad (k \text{ constant})$ $\qquad$ $p = p_0$ when $h = 0$

Using the Law of Exponential Change (p509) we know that $p = p_0\, e^{kh}$. In the problem we are given that $p(0) = 1013$ and $p(20) = 90$.

We will use these initial conditions to find the values of p_0 and k.

$$\begin{aligned} p(0) = 1013 \quad &\Rightarrow \quad p_0\, e^0 = 1013 \\ &\Rightarrow \quad p_0 = 1013 \end{aligned}$$

Thus, $p(h) = 1013e^{kh}$

$$\begin{aligned} p(20) = 90 \quad &\Rightarrow \quad 1013\, e^{20k} = 90 \\ &\Rightarrow \quad e^{20k} = \frac{90}{1013} \\ &\Rightarrow \quad 20k = \ln\left(\frac{90}{1013}\right) \\ &\Rightarrow \quad k = \frac{1}{20}\ln\left(\frac{90}{1013}\right) \approx -0.121 \end{aligned}$$

Thus, $p(h) \approx 1013e^{-0.121h}$

b. $p(50) = 1013\, e^{(-0.121)(50)} \approx 2.389$ milibars

c. $900 = 1013\, e^{-0.121h}$

$\Leftrightarrow \frac{900}{1013} = e^{-0.121h}$

$\Leftrightarrow \ln\left(\frac{900}{1013}\right) = -0.121h$

$\Leftrightarrow h \approx 0.977$ km

6. $\frac{dV}{dt} = -\frac{1}{40}V \quad \Rightarrow \quad V = V_0\, e^{-t/40}$

We want to find t such that $V(t) = 0.1V_0$.

$$\begin{aligned} 0.1V_0 = V_0\, e^{-t/40} \quad &\Rightarrow \quad 0.1 = e^{-t/40} \\ &\Rightarrow \quad \ln(0.1) = -\frac{t}{40} \\ &\Rightarrow \quad t = -40\ln(0.1) \approx 92.1 \text{ sec} \end{aligned}$$

7. We will let the population of the bacteria colony be given by $p(t) = e^{kt}$ where t in measured in hours (we are told $p_0 = 1$).

We know that the p doubles every half hour. Thus $p(0.5) = 2$. We can use this information to find the value of k.

$$\begin{aligned} 2 = e^{0.5k} \quad &\Rightarrow \quad \ln(2) = 0.5k \\ &\Rightarrow \quad k = 2\ln(2) \\ &\Rightarrow \quad k = \ln(4) \end{aligned}$$

Thus we have that $p(t) = e^{\ln(4)t}$. In 24 hours there are $p(24) = e^{\ln(4)\cdot 24} = 4^{24} \approx 2.81475 \times 10^{14}$ bacteria.

21. We know that the temperature (as a function of time) is given by:

$$H(t) = 20 + (90 - 20)\, e^{-kt} = 20 + 70e^{-kt}$$

.

We know that $H(10) = 60$. Using this information, we can solve for the constant k.

$$\begin{aligned} 20 + 70e^{-10k} = 60 \quad &\Rightarrow \quad 70e^{-10k} = 40 \\ &\Rightarrow \quad e^{-10k} = \frac{4}{7} \\ &\Rightarrow \quad -10k = \ln\left(\frac{4}{7}\right) \\ &\Rightarrow \quad k = -\frac{1}{10}\ln\left(\frac{4}{7}\right) \\ &\Rightarrow \quad k \approx 0.05596 \end{aligned}$$

Thus, we have $H(t) = 20 + 70e^{-0.05596t}$.

a.

$$\begin{aligned} 20 + 70e^{-0.05596t} = 35 \quad &\Rightarrow \quad 70e^{-0.05596t} = 15 \\ &\Rightarrow \quad e^{-0.05596t} = \frac{3}{14} \\ &\Rightarrow \quad -0.05596t = \ln\left(\frac{3}{14}\right) \\ &\Rightarrow \quad t \approx 27.5 \text{ min} \end{aligned}$$

We know that it took 10 minutes for the temperature to reach 60°C, so it takes an additional 17.5 minutes for the temperature to reach 35°C.

b. Now we are changing H_S to -15°C.

$$H(t) = -15 + (90 + 15)e^{-0.05596t} = -15 + 105e^{-0.05596t}$$

$$\begin{aligned}
-15+105e^{-0.05596t}=35 \quad &\Rightarrow \quad 105e^{-0.05596t}=50\\
&\Rightarrow \quad e^{-0.05596}=\frac{10}{21}\\
&\Rightarrow \quad -0.05596t=\ln\left(\frac{10}{21}\right)\\
&\Rightarrow \quad t\approx 13.26 \text{ min}
\end{aligned}$$

Section 8.1:

7.

$$\begin{aligned}
\int \frac{1}{\sqrt{x}\,(\sqrt{x}+1)}\,dx &= \int \frac{2}{u}\,du \qquad \left(u=\sqrt{x}+1, \quad du=\frac{1}{2\sqrt{x}}\right)\\
&= 2\ln|u|+C\\
&= 2\ln(\sqrt{x}+1)+C
\end{aligned}$$

36.

$$\begin{aligned}
\int \frac{\ln x}{x+4x\ln^2 x}\,dx &= \int \frac{\ln x}{x\left(1+4\ln^2 x\right)}\,dx\\
&= \int \frac{1}{8u}\,du \qquad \left(u=1+4\ln^2 x, \quad du=\frac{8\ln x}{x}\,dx\right)\\
&= \frac{1}{8}\ln|u|+C\\
&= \frac{1}{8}\ln(1+4\ln^2 x)+C
\end{aligned}$$

39.

$$\begin{aligned}
\int \frac{1}{\sqrt{-t^2+4t-3}}\,dt &= \int \frac{1}{\sqrt{-(t^2-4t+3)}}\,dt\\
&= \int \frac{1}{\sqrt{1-(t^2-4t+4)}}\,dt\\
&= \int \frac{1}{\sqrt{1-(t-2)^2}}\,dt\\
&= \int \frac{1}{\sqrt{1-u^2}}\,du \qquad (u=t-2, \quad du=dt)\\
&= \sin^{-1}(u)+C\\
&= \sin^{-1}(t-2)+C
\end{aligned}$$

42.

$$\begin{aligned}
\int \frac{1}{(x-2)\sqrt{x^2-4x+3}}\,dx &= \int \frac{1}{(x-2)\sqrt{(x^2-4x+4)-1}}\,dx \\
&= \int \frac{1}{(x-2)\sqrt{(x-2)^2-1}}\,dx \\
&= \int \frac{1}{u\sqrt{u^2-1}}\,du \qquad (u = x-2, \quad du = dx) \\
&= \sec^{-1}|u| + C \\
&= \sec^{-1}|x-2| + C \qquad (|x-2| > 1 \text{ for the domain of } \sec^{-1} x)
\end{aligned}$$

43.

$$\begin{aligned}
\int (\sec x + \cot x)^2\,dx &= \int \sec^2 x + 2\sec x \cot x + \cot^2 x\,dx \\
&= \int \sec^2 x + 2\csc x + \cot^2 x\,dx \\
&= \int \sec^2 x + 2\csc x + \left(\csc^2 x - 1\right)\,dx \\
&= \tan x - 2\ln|\csc x + \cot x| - \cot x - x + C
\end{aligned}$$

48.

$$\begin{aligned}
\int \frac{x^2}{x^2+1}\,dx &= \int 1 - \frac{1}{x^2+1}\,dx \\
&= x - \tan^{-1} x + C
\end{aligned}$$

51.

$$\begin{aligned}
\int \frac{4t^3 - t^2 + 16t}{t^2+4}\,dt &= \int 4t - 1 + \frac{4}{t^2+4}\,dt \\
&= 2t^2 - t + 2\tan^{-1}\left(\frac{t}{2}\right) + C
\end{aligned}$$

55.

$$\begin{aligned}\int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}\,dx &= \int_0^{\pi/4} \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\,dx \\ &= \int_0^{\pi/4} \sec^2 x + \sec x \tan x\,dx \\ &= \tan x + \sec x\Big|_0^{\pi/4} \\ &= \left[\tan\left(\frac{\pi}{4}\right) + \sec\left(\frac{\pi}{4}\right)\right] - [\tan(0) + \sec(0)] \\ &= (1+\sqrt{2}) - (0+1) \\ &= \sqrt{2}\end{aligned}$$

61.

$$\begin{aligned}\int \frac{1}{1-\sec x}\,dx &= \int \frac{1}{1-(1/\cos x)}\,dx \\ &= \int \frac{\cos x}{\cos x - 1}\,dx \\ &= \int 1 + \frac{1}{\cos x - 1}\,dx \\ &= \int 1 + \frac{1}{\cos x - 1}\cdot\frac{\cos x + 1}{\cos x + 1}\,dx \\ &= \int 1 + \frac{\cos x + 1}{\cos^2 x - 1}\,dx \\ &= \int 1 - \frac{\cos x + 1}{\sin^2 x}\,dx \\ &= \int 1 - \csc x \cot x - \csc^2 x\,dx \\ &= x + \csc x + \cot x + C\end{aligned}$$

63.

$$\begin{aligned}\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\,dx &= \int_0^{2\pi} \sqrt{\sin^2\left(\frac{x}{2}\right)}\,dx \\ &= \int_0^{2\pi} \left|\sin\left(\frac{x}{2}\right)\right|\,dx \\ &= \int_0^{2\pi} \sin\left(\frac{x}{2}\right)\,dx \qquad \left(\text{since } \sin\left(\tfrac{x}{2}\right) \geq 0 \text{ when } 0 \leq x \leq 2\pi\right) \\ &= -2\cos\left(\frac{x}{2}\right)\Big|_0^{2\pi} \\ &= -2\cos(\pi) + 2\cos(0) \\ &= 4\end{aligned}$$

83. a.

$$\int \cos^3\theta\,d\theta = \int \cos\theta(1-\sin^2\theta)\,d\theta$$
$$= \int \cos\theta - \sin^2\theta\cos\theta\,d\theta$$
$$= \sin\theta - \frac{1}{3}\sin^3\theta + C$$

b.

$$\int \cos^5\theta\,d\theta = \int \cos\theta(1-\sin^2\theta)^2\,d\theta$$
$$= \int \cos\theta(1-2\sin^2\theta+\sin^4\theta)\,d\theta$$
$$= \int \cos\theta - 2\sin^2\theta\cos\theta + \sin^4\theta\cos\theta\,d\theta$$
$$= \sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta + C$$

c. Use the fact that $\int \cos^9\theta\,d\theta = \int \cos\theta(1-\sin^2\theta)^4\,d\theta$.

Section 8.2:

3. $\int t^2\cos t\,dt$

Let $u_1 = t^2$, $du_1 = 2t\,dt$ and $v_1 = \sin t$, $dv_1 = \cos t\,dt$

Let $u_2 = 2t$, $du_2 = 2dt$ and $v_2 = -\cos t$, $dv_2 = \sin t\,dt$

$$\int t^2\cos t\,dt = u_1v_1 - \int v_1\,du_1$$
$$= t^2\sin t - \int 2t\sin t\,dt$$
$$= t^2\sin t - \left[u_2v_2 - \int v_2\,du_2\right]$$
$$= t^2\sin t - \left[-2t\cos t + \int 2\cos t\,dt\right]$$
$$= t^2\sin t + 2t\cos t - 2\sin t + C$$

6. $\int_1^e x^3\ln x\,dx$

Let $u = \ln x, \quad du = \frac{1}{x}\,dx$ and $v = \frac{1}{4}x^4, \quad dv = x^3\,dx$.

$$\begin{aligned}
\int_1^e x^3 \ln x\,dx &= uv|_1^e - \int_1^e v\,du \\
&= \frac{1}{4}x^4 \ln x\bigg|_1^e - \int_1^e \frac{1}{4}x^4 \cdot \frac{1}{x}\,dx \\
&= \frac{1}{4}x^4 \ln x\bigg|_1^e - \frac{1}{4}\int_1^e x^3\,dx \\
&= \frac{1}{4}x^4 \ln x\bigg|_1^e - \frac{1}{16}x^4\bigg|_1^e \\
&= \left(\frac{e^4}{4} - 0\right) - \left(\frac{e^4}{16} - \frac{1}{16}\right) \\
&= \frac{3e^4+1}{16}
\end{aligned}$$

16. $\int t^2 e^{4t}\,dt$

Let $u_1 = t^2, \quad du_1 = 2t\,dt$ and $v_1 = \frac{1}{4}e^{4t}, \quad dv_1 = e^{4t}\,dt$.

Let $u_2 = 2t, \quad du_2 = 2dt$ and $v_2 = \frac{1}{16}e^{4t}, \quad dv_2 = \frac{1}{4}e^{4t}\,dt$.

$$\begin{aligned}
\int t^2 e^{4t}\,dt &= u_1 v_1 - \int v_1\,du_1 \\
&= \frac{1}{4}t^2 e^{4t} - \int \frac{1}{4}2te^{4t}\,dt \\
&= \frac{1}{4}t^2 e^{4t} - \left[\frac{1}{8}te^{4t} - \int \frac{1}{8}e^{4t}\,dt\right] \\
&= \frac{1}{4}t^2 e^{4t} - \frac{1}{8}te^{4t} + \frac{1}{32}e^{4t} + C
\end{aligned}$$

21. $\int e^{\theta} \sin\theta\,d\theta$

Let $u_1 = \sin\theta, \quad du_1 = \cos\theta\,d\theta$ and $v_1 = e^{\theta}, \quad dv_1 = e^{\theta}\,d\theta$.

Let $u_2 = \cos\theta$, $\quad du_2 = -\sin\theta\, d\theta$ and $v_2 = e^\theta$, $\quad dv_2 = e^\theta\, d\theta$.

$$\begin{aligned}
\int e^\theta \sin\theta\, d\theta &= u_1 v_1 - \int v_1\, du_1 \\
&= e^\theta \sin\theta - \int e^\theta \cos\theta\, d\theta \\
&= e^\theta \sin\theta - \left[u_2 v_2 - \int v_2\, du_2\right] \\
&= e^\theta \sin\theta - \left[e^\theta \cos\theta + \int e^\theta \sin\theta\, d\theta\right] \\
&= e^\theta \sin\theta - e^\theta \cos\theta - \int e^\theta \sin\theta\, d\theta + C'
\end{aligned}$$

We can use this to solve for $\int e^\theta \sin\theta\, d\theta$.

$$\begin{aligned}
\int e^\theta \sin\theta\, d\theta &= e^\theta \sin\theta - e^\theta \cos\theta - \int e^\theta \sin\theta\, d\theta + C' \\
\Leftrightarrow 2\int e^\theta \sin\theta\, d\theta &= e^\theta \sin\theta - e^\theta \cos\theta + C' \\
\Leftrightarrow \int e^\theta \sin\theta\, d\theta &= \frac{e^\theta \sin\theta - e^\theta \cos\theta}{2} + C
\end{aligned}$$

27. $\displaystyle\int_0^{\pi/3} x \tan^2 x\, dx$

Let $u = x$, $\quad du = dx$ and $dv = \tan^2 x$, $\quad v = \int \tan^2 x\, dx = \int \sec^2 x - 1\, dx = \tan x - x$.

$$\begin{aligned}
\int_0^{\pi/3} x \tan^2 x\, dx &= uv\Big|_0^{\pi/3} - \int_0^{\pi/3} v\, du \\
&= \left(x\tan x - x^2\right)\Big|_0^{\pi/3} - \int_0^{\pi/3} \left(\tan x - x\right)\, dx \\
&= \left(x\tan x - x^2\right)\Big|_0^{\pi/3} - \left(\ln|\sec x| - \frac{1}{2}x^2\right)\Bigg|_0^{\pi/3} \\
&= \left(\frac{\pi\sqrt{3}}{3} - \frac{\pi^2}{9}\right) - \left(\ln(2) - \frac{\pi^2}{18}\right) \\
&= \frac{\pi\sqrt{3}}{3} - \frac{\pi^2}{18} - \ln(2)
\end{aligned}$$

30. $\int z(\ln z)^2\,dz$

Let $w = \ln z$, $\quad dw = \frac{1}{z}\,dz$. Then we get that $z = e^w$.

$$\int z(\ln z)^2\,dz = \int \frac{z^2(\ln z)^2}{z}\,dz = \int w^2 e^{2w}\,dw$$

Let $u_1 = w^2$, $\quad du_1 = 2w\,dw$ and $v_1 = \frac{1}{2}e^{2w}$, $\quad dv_1 = e^{2w}\,dw$.

Let $u_2 = w$, $\quad du_2 = dw$ and $v_2 = \frac{1}{2}e^{2w}$, $\quad dv_2 = e^{2w}\,dw$.

$$\begin{aligned}
\int w^2 e^{2w}\,dw &= u_1 v_1 - \int v_1\,du_1 \\
&= \frac{1}{2}w^2e^{2w} - \int we^{2w}\,dw \\
&= \frac{1}{2}w^2e^{2w} - \left[u_2v_2 - \int v_2\,du_2\right] \\
&= \frac{1}{2}w^2e^{2w} - \left[\frac{1}{2}we^{2w} - \int \frac{1}{2}e^{2w}\,dw\right] \\
&= \frac{1}{2}w^2e^{2w} - \frac{1}{2}we^{2w} + \frac{1}{4}e^{2w} + C
\end{aligned}$$

We now need to go back and substitute z back into the solution.

$$\int z(\ln z)^2\,dz = \frac{1}{2}z^2(\ln z)^2 - \frac{1}{2}z^2\ln z + \frac{1}{4}z^2 + C$$

39. $\int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1}\sin x\,dx$

Let $u = x^n$, $\quad du = nx^{n-1}\,dx$ and $v = \sin x$, $\quad dv = \cos x\,dx$.

$$\begin{aligned}
\int x^n \cos x\,dx &= uv - \int v\,du \\
&= x^n \sin x - \int nx^{n-1}\sin x\,dx \\
&= x^n \sin x - n\int x^{n-1}\sin x\,dx
\end{aligned}$$

42. $\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

Let $u = (\ln x)^n$, $du = \frac{n(\ln x)^{n-1}}{x}\,dx$ and $v = x$, $dv = dx$.

$$\begin{aligned}\int (\ln x)^n\,dx &= uv - \int v\,du \\ &= x(\ln x)^n - \int x \cdot \frac{n(\ln x)^{n-1}}{x}\,dx \\ &= x(\ln x)^n - n\int (\ln x)^{n-1}\,dx\end{aligned}$$

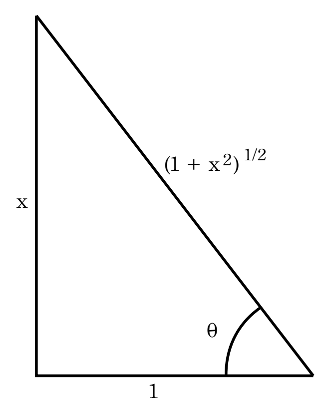
43.

$$\begin{aligned}\int \sin^{-1} x\,dx &= x\sin^{-1} x - \int \sin(y)\,dy \\ &= x\sin^{-1} x + \cos y + C \\ &= x\sin^{-1} x + \cos\left(\sin^{-1} x\right) + C\end{aligned}$$

48. a. $\displaystyle\int \tan^{-1} x\,dx = x\tan^{-1} x - \ln\left[\sec\left(\tan^{-1} x\right)\right] + C$

b. $\displaystyle\int \tan^{-1} x\,dx = x\tan^{-1} x - \ln\left(\sqrt{1+x^2}\right) + C$

These two integrations are correct. We want to justify that $\sec\left(\tan^{-1} x\right) = \sqrt{1+x^2}$. We will begin with the left-hand side. If we let $\theta = \tan^{-1} x$, we are lead to the following right triangle:



By the triangle we see that $\sec\left(\tan^{-1}\theta\right) = \sec(\theta) = \frac{hypotenuse}{adjacent} = \sqrt{1+x^2}$.