Math 21B-B - Homework Set 8

Section 8.3:

11. $\displaystyle\int \frac{x+4}{x^2+5x-6}\,dx = \int \frac{x+4}{(x+6)(x-1)}\,dx$

$$\frac{x+4}{(x+6)(x-1)} = \frac{A}{x+6} + \frac{B}{x-1} \qquad \Leftrightarrow \qquad x+4 = (A+B)x + (-A+6B)$$
$$\Leftrightarrow \qquad \begin{cases} 1 = A+B \\ 4 = -A+6B \end{cases}$$
$$\Leftrightarrow \qquad A = \frac{2}{7}, \quad B = \frac{5}{7}$$

Therefore, going back to our integral we get:

$$\int \frac{x+4}{x^2+5x-6}\,dx = \int \left(\frac{2}{7}\cdot\frac{1}{x+6} + \frac{5}{7}\cdot\frac{1}{x-1}\right)dx$$
$$= \frac{2}{7}\ln|x+6| + \frac{5}{7}\ln|x-1| + C$$

16. $\displaystyle\int \frac{x+3}{2x^3-8x}\,dx = \int \frac{x+3}{2x(x-2)(x+2)}\,dx$

$$\frac{x+3}{2x(x-2)(x+2)} = \frac{A}{2x} + \frac{B}{x-2} + \frac{C}{x+2} \qquad \Leftrightarrow \qquad x+3 = (A+2B+2C)x^2 + (4C-4B)x - 4A$$
$$\Leftrightarrow \qquad \begin{cases} 0 = A+2B+2C \\ 1 = 4C-4B \\ 3 = -4A \end{cases}$$
$$\Leftrightarrow \qquad A = -\frac{3}{4}, \quad B = \frac{1}{16}, \quad C = \frac{5}{16}$$

Therefore, going back to our integral we get:

$$\int \frac{x+3}{2x^3-8x}\,dx = \int \left(-\frac{3}{4}\cdot\frac{1}{2x} + \frac{1}{16}\cdot\frac{1}{x-2} + \frac{5}{16}\cdot\frac{1}{x+2}\right)dx$$
$$= -\frac{3}{8}\ln|x| + \frac{1}{16}\ln|x-2| + \frac{5}{16}\ln|x+2| + C$$

17. $\displaystyle\int_0^1 \frac{x^3}{x^2+2x+1}\,dx = \int_0^1 \left(x-2+\frac{3x+2}{x^2+2x+1}\right) dx \qquad (\textit{by long division})$

$$\frac{3x+2}{x^2+2x+1} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \qquad \Leftrightarrow \qquad 3x+2 = Ax + (A+B)$$

$$\Leftrightarrow \qquad \begin{cases} 3 = A \\ 2 = A+B \end{cases}$$

$$\Leftrightarrow A = 3, \quad B = -1$$

Therefore, going back to our integral we get:

$$\begin{aligned} \int_0^1 \frac{x^3}{x^2+2x+1}\,dx &= \int_0^1 \left(x-2+\frac{3}{x+1}-\frac{1}{(x+1)^2}\right) dx \\ &= \left(\frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1}\right)\Big|_0^1 \\ &= \left(\frac{1}{2} - 2 + 3\ln 2 + \frac{1}{2}\right) - (1) \\ &= 3\ln 2 - 2 \end{aligned}$$

21. $\displaystyle\int_0^1 \frac{dx}{(x+1)(x^2+1)}$

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \qquad \Leftrightarrow \qquad 1 = (A+B)x^2 + (B+C)x + (C+A)$$

$$\Leftrightarrow \qquad \begin{cases} 0 = A+B \\ 0 = B+C \\ 1 = C+A \end{cases}$$

$$\Leftrightarrow \qquad A = \frac{1}{2}, \quad B = -\frac{1}{2}, \quad C = \frac{1}{2}$$

Therefore, going back to our integral we get:

$$\begin{aligned} \int_0^1 \frac{dx}{(x+1)(x^2+1)} &= \int_0^1 \left(\frac{1}{2}\cdot\frac{1}{x+1} + \frac{1}{2}\cdot\frac{1-x}{x^2+1}\right) dx \\ &= \frac{1}{2}\int_0^1 \left(\frac{1}{x+1} + \frac{1}{x^2+1} - \frac{x}{x^2+1}\right) dx \\ &= \frac{1}{2}\left(\ln|x+1| + \tan^{-1} x - \frac{1}{2}\ln|x^2+1|\right)\Big|_0^1 \\ &= \frac{1}{2}\left(\ln 2 + \frac{\pi}{4} - \frac{\ln 2}{2}\right) \\ &= \frac{1}{4}\ln 2 + \frac{\pi}{8} \end{aligned}$$

22. $\displaystyle\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dt$

$$\frac{3t^2+t+4}{t(t^2+1)} = \frac{A}{t} + \frac{Bx+C}{t^2+1} \qquad \Leftrightarrow \qquad 3t^2+t+4 = (A+B)t^2 + Ct + A$$

$$\Leftrightarrow \qquad \begin{cases} 3 = A+B \\ 1 = C \\ 4 = A \end{cases}$$

$$\Leftrightarrow \qquad A = 4, \quad B = -1, \quad C = 1$$

Therefore, going back to our integral we get:

$$\begin{aligned}
\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dx &= \int_1^{\sqrt{3}} \left(\frac{4}{t} + \frac{1-t}{t^2+1}\right) dt \\
&= \int_1^{\sqrt{3}} \left(\frac{4}{t} + \frac{1}{t^2+1} - \frac{t}{t^2+1}\right) dt \\
&= \left(4\ln|t| + \tan^{-1} t - \frac{1}{2}\ln|t^2+1|\right)\Bigg|_1^{\sqrt{3}} \\
&= \left(4\ln(\sqrt{3}) + \frac{\pi}{3} - \frac{1}{2}\ln 4\right) - \left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right) \\
&= 2\ln 3 + \frac{\pi}{3} - \ln 2 - \frac{\pi}{4} + \frac{1}{2}\ln 2 \\
&= 2\ln 3 - \frac{1}{2}\ln 2 + \frac{\pi}{12}
\end{aligned}$$

30. $\displaystyle\int \frac{x^4}{x^2-1}\,dx = \int \left(x^2+1+\frac{1}{x^2-1}\right) dx$

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} \qquad \Leftrightarrow \qquad 1 = (A+B)x + (A-B)$$

$$\Leftrightarrow \qquad \begin{cases} 0 = A+B \\ 1 = A-B \end{cases}$$

$$\Leftarrow \qquad A = \frac{1}{2}, \quad B = -\frac{1}{2}$$

Therefore, going back to our integral we get:

$$\begin{aligned}
\int \frac{x^4}{x^2-1}\,dx &= \int \left(x^2+1+\frac{1}{2}\cdot\frac{1}{x-1} - \frac{1}{2}\cdot\frac{1}{x+1}\right) dx \\
&= \frac{1}{3}x^3 + x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C \\
&= \frac{1}{3}x^3 + x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C
\end{aligned}$$

Section 8.4:

1.

$$\begin{aligned}
\int_0^{\pi/2} \sin^5 x\,dx &= \int_0^{\pi/2} \sin x \left(1-\cos^2 x\right)^2\,dx \\
&= \int_0^{\pi/2} \left(\sin x - 2\cos^2 x \sin x + \cos^4 x \sin x\right)\,dx \\
&= \left(-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x\right)\Big|_0^{\pi/2} \\
&= -\left(-1+\frac{2}{3}-\frac{1}{5}\right) \\
&= \frac{8}{15}
\end{aligned}$$

8.

$$\begin{aligned}
\int_0^1 8\cos^4(2\pi x)\,dx &= 8\int_0^1 \left(\frac{\cos(4\pi x)+1}{2}\right)^2\,dx \\
&= 8\int_0^1 \frac{\cos^2(4\pi x)+2\cos(4\pi x)+1}{4}\,dx \\
&= 2\int_0^1 \left(\frac{\cos(8\pi x)+1}{2}+2\cos(4\pi x)+1\right)\,dx \\
&= \int_0^1 \left[\cos(8\pi x)+4\cos(4\pi x)+3\right]\,dx \\
&= \left(\frac{1}{8\pi}\sin(8\pi x)+\frac{1}{\pi}\sin(4\pi x)+3x\right)\Big|_0^1 \\
&= 3
\end{aligned}$$

15.

$$\begin{aligned}
\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\,dx &= \int_0^{2\pi} \sqrt{\sin^2\left(\frac{x}{2}\right)}\,dx \\
&= \int_0^{2\pi} \left|\sin\left(\frac{x}{2}\right)\right|\,dx \\
&= \int_0^{2\pi} \sin\left(\frac{x}{2}\right)\,dx \qquad \left(\sin\left(\frac{x}{2}\right)\geq 0,\quad 0\leq x\leq 2\pi\right) \\
&= -2\cos\left(\frac{x}{2}\right)\Big|_0^{2\pi} \\
&= 2-(-2) \\
&= 4
\end{aligned}$$

20.

$$\begin{aligned}
\int_{-\pi/4}^{\pi/4} \sqrt{\sec^2 x - 1}\, dx &= \int_{-\pi/4}^{\pi/4} \sqrt{\tan^2 x}\, dx \\
&= \int_{-\pi/4}^{\pi/4} |\tan x|\, dx \\
&= 2\int_{0}^{\pi/4} \tan x\, dx \\
&= \left(2\ln|\sec x|\right)\Big|_0^{\pi/4} \\
&= 2\ln(\sqrt{2}) \\
&= \ln 2
\end{aligned}$$

25.

$$\begin{aligned}
\int_{0}^{\pi/4} \sec^4 x\, dx &= \int_{0}^{\pi/4} \sec^2 x(\tan^2 x + 1)\, dx \\
&= \int_{0}^{\pi/4} (\sec^2 x \tan^2 x + \sec^2 x)\, dx \\
&= \left(\frac{1}{3}\tan^3 x + \tan x\right)\Big|_0^{\pi/4} \\
&= \frac{1}{3} + 1 \\
&= \frac{4}{3}
\end{aligned}$$

30.

$$\begin{aligned}
\int_{-\pi/4}^{\pi/4} 6\tan^4 x\, dx &= 6\int_{-\pi/4}^{\pi/4} \tan^2 x(\sec^2 - 1)\, dx \\
&= 6\int_{-\pi/4}^{\pi/4} (\tan^2 x \sec^2 x - \tan^2 x)\, dx \\
&= 6\int_{-\pi/4}^{\pi/4} (\tan^2 x \sec^2 x - \sec^2 x + 1)\, dx \\
&= 6\left(\frac{1}{3}\tan^3 x - \tan x + x\right)\Big|_{-\pi/4}^{\pi/4} \\
&= 6\left[\left(\frac{1}{3} - 1 + \frac{\pi}{4}\right) - \left(-\frac{1}{3} + 1 - \frac{\pi}{4}\right)\right] \\
&= 3\pi - 8
\end{aligned}$$

33.

$$\int_{-\pi}^{0} \sin(3x)\cos(2x)\,dx = \int_{-\pi}^{0} \left(\frac{1}{2}\sin x + \frac{1}{2}\sin(5x)\right)\,dx \qquad \textit{(see p569)}$$

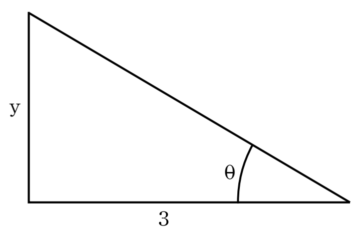
$$= \left(-\frac{1}{2}\cos x - \frac{1}{10}\cos(5x)\right)\Big|_{-\pi}^{0}$$

$$= \left(-\frac{1}{2} - \frac{1}{10}\right) - \left(\frac{1}{2} + \frac{1}{10}\right)$$

$$= -\frac{6}{5}$$

Section 8.5:

1. $\int \frac{1}{\sqrt{9+y^2}}\,dy$



Let $y = 3\tan\theta$, $\quad dy = 3\sec^2\theta\,d\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

$$\int \frac{1}{\sqrt{9+y^2}}\,dy = \int \frac{3\sec^2\theta}{\sqrt{9+9\tan^2\theta}}\,d\theta$$

$$= \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\sec\left(\tan^{-1}\left(\frac{y}{3}\right)\right) + \tan\left(\tan^{-1}\left(\frac{y}{3}\right)\right)\right| + C$$

$$= \ln\left|\frac{\sqrt{9+y^2}}{3} + \frac{y}{3}\right| + C \qquad \text{or} \qquad \ln\left|\sqrt{9+y^2} + y\right| + C'$$

4. $\int_0^2 \frac{dx}{8+2x^2}$

Let $x = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

$$\begin{aligned}
\int_0^2 \frac{dx}{8+2x^2} &= \int_{\tan^{-1}(0)}^{\tan^{-1}(1)} \frac{2\sec^2\theta}{8+8\tan^2\theta}\, d\theta \\
&= \int_0^{\pi/4} \frac{2\sec^2\theta}{8\sec^2\theta}\, d\theta \\
&= \int_0^{\pi/4} \frac{1}{4}\, d\theta \\
&= \left.\frac{\theta}{4}\right|_0^{\pi/4} \\
&= \frac{\pi}{16}
\end{aligned}$$

OR

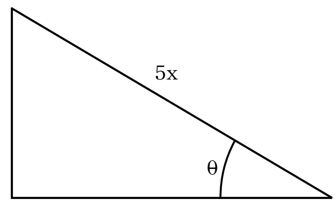
$$\begin{aligned}
\int_0^2 \frac{dx}{8+2x^2} &= \frac{1}{8}\int_0^2 \frac{dx}{1+\left(\frac{x}{2}\right)^2}\, dx \\
&= \frac{1}{8}\cdot 2\tan^{-1}\left.\left(\frac{x}{2}\right)\right|_0^2 \\
&= \frac{\pi}{16}
\end{aligned}$$

5. $\displaystyle\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}}$

Let $x = 3\sin\theta$, $dx = 3\cos\theta\, d\theta$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

$$\begin{aligned}
\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}} &= \int_0^{\pi/6} \frac{3\cos\theta}{\sqrt{9-9\sin^2\theta}}\, d\theta \\
&= \int_0^{\pi/6} \frac{3\cos\theta}{3\cos\theta}\, d\theta \\
&= \int_0^{\pi/6} d\theta \\
&= \theta|_0^{\pi/6} \\
&= \frac{\pi}{6}
\end{aligned}$$

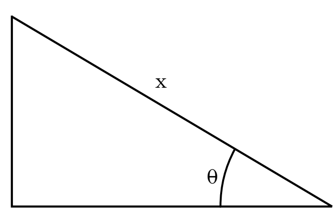
10. $\displaystyle\int \frac{5}{\sqrt{25x^2-9}}\, dx$ for $x > \frac{3}{5}$



Let $x = \frac{3}{5}\sec\theta$, $\quad dx = \frac{3}{5}\sec\theta\tan\theta\,d\theta$ $\;$ for $0 \le \theta < \frac{\pi}{2}$

$$
\begin{aligned}
\int \frac{5}{\sqrt{25x^2-9}}\,dx &= \int \frac{5\cdot(3/5)\sec\theta\tan\theta}{\sqrt{25\cdot(9/25)\sec^2\theta-9}}\,d\theta \\
&= \int \frac{3\sec\theta\tan\theta}{3\sqrt{\sec^2\theta-1}}\,d\theta \\
&= \int \frac{\sec\theta\tan\theta}{\sqrt{\tan^2\theta}}\,d\theta \\
&= \int \sec\theta\,d\theta \\
&= \ln|\sec\theta+\tan\theta| + C \\
&= \ln\left|\sec\left(\sec^{-1}\left(\frac{5x}{3}\right)\right) + \tan\left(\sec^{-1}\left(\frac{5x}{3}\right)\right)\right| + C \\
&= \ln\left|\frac{5}{3}x + \frac{\sqrt{25x^2-9}}{3}\right| + C \qquad \text{or} \qquad \ln\left|5x+\sqrt{25x^2-9}\right| + C'
\end{aligned}
$$

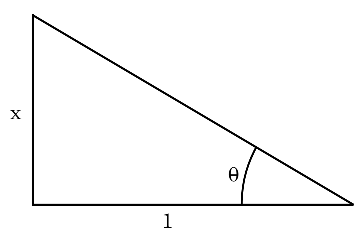
13. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-1}}\,dx \quad$ for $x > 1$



Let $x = \sec\theta$, $\quad dx = \sec\theta\tan\theta\, d\theta \;$ for $0 \le \theta < \frac{\pi}{2}$.

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2-1}}\, dx &= \int \frac{\sec\theta\tan\theta}{\sec^2\theta\sqrt{\sec^2\theta - 1}}\, d\theta \\
&= \int \frac{\sec\theta\tan\theta}{\sec^2\theta|\tan\theta|}\, d\theta \\
&= \int \frac{\sec\theta\tan\theta}{\sec^2\theta\tan\theta}\, d\theta \qquad \left(\tan\theta \ge 0, \quad 0 \le \theta < \frac{\pi}{2}\right) \\
&= \int \cos\theta\, d\theta \\
&= \sin\theta + C \\
&= \sin\left(\sec^{-1}(x)\right) + C \\
&= \frac{\sqrt{x^2-1}}{x} + C
\end{aligned}$$

16. $\displaystyle\int \frac{1}{x^2\sqrt{x^2+1}}\, dx$



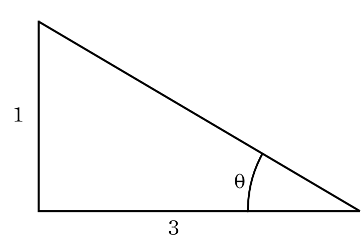
Let $x = \tan\theta$, $\quad dx = \sec^2\theta\, d\theta \quad$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

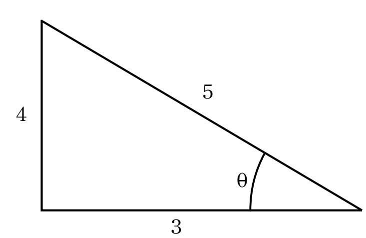
$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2+1}}\,dx &= \int \frac{\sec^2\theta}{\tan^2\theta\sqrt{\tan^2\theta+1}}\,d\theta \\
&= \int \frac{\sec^2\theta}{\tan^2\theta\sqrt{\sec^2\theta}}\,d\theta \\
&= \int \frac{\sec^2\theta}{\tan^2\theta\sec\theta}\,d\theta \qquad \left(\sec\theta > 1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right) \\
&= \int \frac{\sec\theta}{\tan^2\theta}\,d\theta \\
&= \int \frac{\cos\theta}{\sin^2\theta}\,d\theta \\
&= -\frac{1}{\sin\theta} + C \\
&= -\frac{1}{\sin\left(\tan^{-1}(x)\right)} + C \\
&= -\frac{\sqrt{x^2+1}}{x} + C
\end{aligned}$$

29. $\displaystyle\int_0^{\ln 4} \frac{e^t}{\sqrt{e^{2t}+9}}\,dt$

Firstly, let $x = e^t$, $\quad dx = e^t\,dt$.

$$\int_0^{\ln 4} \frac{e^t}{\sqrt{e^{2t}+9}}\,dt = \int_1^4 \frac{1}{\sqrt{x^2+9}}\,dx$$





Now, let $x = 3\tan\theta, \quad dx = 3\sec^2\theta\,d\theta$ for $\tan^{-1}\left(\frac{1}{3}\right) < \theta < \tan^{-1}\left(\frac{4}{3}\right)$.

$$\begin{aligned}
\int_1^4 \frac{1}{\sqrt{x^2+9}}\,dx &= \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \frac{3\sec^2\theta}{\sqrt{9\tan^2\theta+9}}\,d\theta \\
&= \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \frac{3\sec^2\theta}{3\sec\theta}\,d\theta \qquad \left(\sec\theta > 0, \quad \tan^{-1}\left(\frac{1}{3}\right) \le \theta \le \tan^{-1}\left(\frac{4}{3}\right)\right) \\
&= \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \sec\theta\,d\theta \\
&= \ln|\sec\theta + \tan\theta|\Big|_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \\
&= \ln\left|\sec\left(\tan^{-1}(4/3)\right) + \tan\left(\tan^{-1}(4/3)\right)\right| - \ln\left|\sec\left(\tan^{-1}(1/3)\right) + \tan\left(\tan^{-1}(1/3)\right)\right| \\
&= \ln\left(\frac{5}{3} + \frac{4}{3}\right) - \ln\left(\frac{\sqrt{10}}{3} + \frac{1}{3}\right) \\
&= \ln 3 - \ln\left(\sqrt{10}+1\right) + \ln 3 \\
&= 2\ln 3 - \ln\left(\sqrt{10}+1\right)
\end{aligned}$$

33. $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx$

Let $x = \sec\theta, \quad dx = \sec\theta\tan\theta\,d\theta$ for $0 < \theta < \frac{\pi}{2}$

$$\begin{aligned}
\int \frac{1}{x\sqrt{x^2-1}}\,dx &= \int \frac{\sec\theta\tan\theta}{\sec\theta\sqrt{\sec^2\theta-1}}\,d\theta \\
&= \int \frac{\sec\theta\tan\theta}{\sec\theta\sqrt{\tan^2\theta}}\,d\theta \\
&= \int 1\,d\theta \\
&= \theta + C \\
&= \sec^{-1} x + C
\end{aligned}$$

36. $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx$

Let $x = \sin\theta, \quad dx = \cos\theta\, d\theta \quad \text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\begin{aligned}
\int \frac{1}{\sqrt{1-x^2}}\, dx &= \int fraccos\theta \sqrt{1-\sin^2\theta}\, d\theta \\
&= \int \frac{\cos\theta}{\cos\theta}\, d\theta \qquad \left(\cos\theta > 0, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right) \\
&= \int 1\, d\theta \\
&= \theta + C \\
&= \sin^{-1} x + C
\end{aligned}$$

43.

$$\begin{aligned}
\int \frac{1}{1-\sin x}\, dx &= \int \frac{\frac{2dz}{1+z^2}}{1 - \frac{2z}{1+z^2}} \\
&= \int \frac{\left(\frac{2dz}{1+z^2}\right)}{\left(\frac{1+z^2-2z}{1+z^2}\right)}\, dz \\
&= \int \frac{2}{1-2z+z^2}\, dz \\
&= \int \frac{2}{(1-z)^2}\, dz \\
&= \frac{2}{1-z} + C \\
&= \frac{2}{1-\tan\left(\frac{x}{2}\right)} + C
\end{aligned}$$

51.

$$\begin{aligned}
\int \sec\theta\, d\theta &= \int \frac{1+z^2}{1-z^2} \cdot \frac{2}{1+z^2}\, dz \\
&= \int \frac{2}{1-z^2}\, dz \\
&= \int \left(\frac{1}{1-z} + \frac{1}{1+z}\right) dz \\
&= -\ln|1-z| + \ln|z+1| + C \\
&= \ln\left|\frac{1+z}{1-z}\right| + C \\
&= \ln\left|\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right| + C
\end{aligned}$$

Section 8.8:

3.

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}}\,dx &= \lim_{b\to 0^+}\int_b^1 \frac{1}{\sqrt{x}}\,dx\\ &= \lim_{b\to 0^+}\ 2\sqrt{x}\Big|_b^1\\ &= \lim_{b\to 0^+}\ \left(2-2\sqrt{b}\right)\\ &= 2\end{aligned}$$

7.

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx &= \lim_{b\to 1^-}\int_0^b \frac{1}{\sqrt{1-x^2}}\,dx\\ &= \lim_{b\to 1^-}\ \sin^{-1}x\Big|_0^b\\ &= \lim_{b\to 1^-}\ \left(\sin^{-1}(b)-\sin^{-1}(0)\right)\\ &= \frac{\pi}{2}\end{aligned}$$

18.

$$\begin{aligned}\int_1^\infty \frac{1}{x\sqrt{x^2-1}}\,dx &= \lim_{b\to 1^+}\int_b^2 \frac{1}{x\sqrt{x^2-1}}\,dx + \lim_{c\to\infty}\int_2^\infty \frac{1}{x\sqrt{x^2-1}}\,dx\\ &= \lim_{b\to 1^+}\ \sec^{-1}x\Big|_b^2 + \lim_{c\to\infty}\ \sec^{-1}x\Big|_2^c\\ &= \lim_{b\to 1^+}\left[\sec^{-1}(2)-\sec^{-1}(b)\right] + \lim_{c\to\infty}\left[\sec^{-1}(c)-\sec^{-1}(2)\right]\\ &= \frac{\pi}{3}-0+\frac{\pi}{2}-\frac{\pi}{3}\\ &= \frac{\pi}{2}\end{aligned}$$

22. $\displaystyle\int_0^\infty 2e^{-\theta}\sin\theta\,d\theta$

We will first consider the indefinite integral $\displaystyle\int 2e^{-\theta}\sin\theta\,d\theta$.

Let $u_1 = \sin\theta,\quad du_1 = \cos\theta\,d\theta$ and $v_1 = -2e^{-\theta},\quad dv_1 = 2e^{-\theta}\,d\theta$.

Let $u_2 = \cos\theta, \quad du_2 = -\sin\theta\, d\theta$ and $v_2 = -2e^{-\theta}, \quad dv_2 = 2e^{-\theta}\, d\theta$.

$$\begin{aligned}
\int 2e^{-\theta}\sin\theta\, d\theta &= u_1v_1 - \int v_1\, du_1 \\
&= -2e^{-\theta}\sin\theta + \int 2e^{-\theta}\sin\theta\, d\theta \\
&= -2e^{-\theta}\sin\theta + u_2v_2 - \int v_2\, du_2 \\
&= -2e^{-\theta}\sin\theta - 2e^{-\theta}\cos\theta - \int 2e^{-\theta}\sin\theta\, d\theta
\end{aligned}$$

Let $I = \displaystyle\int 2e^{-\theta}\sin\theta\, d\theta$. Solving for I we get:

$$\begin{aligned}
I = -2e^{-\theta}\sin\theta - 2e^{-\theta}\cos\theta - I \qquad &\Leftrightarrow \qquad 2I = -2e^{-\theta}\sin\theta - 2e^{-\theta}\cos\theta \\
&\Leftrightarrow \qquad I = -e^{-\theta}\sin\theta - e^{-\theta}\cos\theta
\end{aligned}$$

So we have that $\displaystyle\int 2e^{-\theta}\sin\theta\, d\theta = -e^{-\theta}\sin\theta - e^{-\theta}\cos\theta + C$. Now let's consider the improper integral $\displaystyle\int_0^\infty 2e^{-\theta}\sin\theta\, d\theta$.

$$\begin{aligned}
\int_0^\infty 2e^{-\theta}\sin\theta\, d\theta &= \lim_{b\to\infty}\int_0^b 2e^{-\theta}\sin\theta\, d\theta \\
&= \lim_{b\to\infty} \left(-e^{-\theta}\sin\theta - e^{-\theta}\cos\theta\right)\Big|_0^b \\
&= \lim_{b\to\infty} [(-e^{-b}\sin b - e^{-b}\cos b) - (-1)] \\
&= 1
\end{aligned}$$

24.

$$\begin{aligned}
\int_{-\infty}^{\infty} 2xe^{-x^2}\, dx &= \lim_{b\to-\infty}\int_b^0 2xe^{-x^2}\, dx + \lim_{c\to\infty}\int_0^c 2xe^{-x^2}\, dx \\
&= \lim_{b\to-\infty} -e^{-x^2}\Big|_b^0 + \lim_{c\to\infty} -e^{-x^2}\Big|_0^c \\
&= \lim_{b\to-\infty}\left(-1 + e^{-b^2}\right) + \lim_{c\to\infty}\left(-e^{-c^2} + 1\right) \\
&= 0
\end{aligned}$$

25.

$$\begin{aligned}
\int_0^1 x\ln x\,dx &= \lim_{b\to 0^+}\int_b^1 x\ln x\,dx \\
&= \lim_{b\to 0^+}\left[\frac{1}{2}x^2\ln x\Big|_b^1 - \int_b^1 \frac{1}{2}x\,dx\right] && \textit{(Integration by Parts)} \\
&= \lim_{b\to 0^+}\left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right)\Big|_b^1 \\
&= \lim_{b\to 0^+}\left[\left(-\frac{1}{4}\right) - \left(\frac{1}{2}b^2\ln b - \frac{1}{4}b^2\right)\right] \\
&= -\frac{1}{4} - \frac{1}{2}\left[\lim_{b\to 0^+}\frac{\ln b}{\left(\frac{1}{b^2}\right)}\right] \\
&= -\frac{1}{4} - \frac{1}{2}\left[\lim_{b\to 0^+}\frac{\left(\frac{1}{b}\right)}{\left(\frac{-2}{b^3}\right)}\right] && \textit{(l'Hopital's Rule)} \\
&= -\frac{1}{4} + \frac{1}{4}\left[\lim_{b\to 0^+} b^2\right] \\
&= -\frac{1}{4}
\end{aligned}$$

31.

$$\begin{aligned}
\int_{-1}^4 \frac{1}{\sqrt{|x|}}\,dx &= \lim_{b\to 0^-}\int_{-1}^b \frac{1}{\sqrt{|x|}}\,dx + \lim_{c\to 0^+}\int_c^4 \frac{1}{\sqrt{|x|}}\,dx \\
&= \lim_{b\to 0^-}\int_{-1}^b \frac{1}{\sqrt{-x}}\,dx + \lim_{c\to 0^+}\int_c^4 \frac{1}{\sqrt{x}}\,dx \\
&= \lim_{b\to 0^-} -2\sqrt{-x}\Big|_{-1}^b + \lim_{c\to 0^+} 2\sqrt{x}\Big|_c^4 \\
&= \lim_{b\to 0^-}\left(-2\sqrt{-b}+2\right) + \lim_{c\to 0^+}\left(4-2\sqrt{c}\right) \\
&= 6
\end{aligned}$$

65. a. • If $p \neq 1$:

$$\begin{aligned}
\int_1^2 \frac{1}{x(\ln x)^p}\,dx &= \lim_{b\to 1^+}\int_b^2 \frac{1}{x(\ln x)^p}\,dx \\
&= \lim_{b\to 1^+}\frac{(\ln x)^{1-p}}{1-p}\Big|_b^2 \\
&= \lim_{b\to 1^+}\left[\frac{(\ln 2)^{1-p}}{1-p} - \frac{(\ln b)^{1-p}}{1-p}\right]
\end{aligned}$$

We know that $\lim_{b\to 1^+} (\ln b) = 0$. Thus we need $1 - p > 0$ in order for our limit to converge.

- If $p = 1$:

$$\begin{aligned}\int_1^2 \frac{1}{x\ln x}\,dx &= \lim_{b\to 1^+} \int_b^2 \frac{1}{x\ln x}\,dx \\ &= \lim_{b\to 1^+} \ln(\ln x)\Big|_b^2 \\ &= \lim_{b\to 1^+} [\ln(\ln 2) - \ln(\ln b)] \\ &\text{diverges to } -\infty\end{aligned}$$

Therefore we get that the integral converges for $p < 1$.

b. - If $p \neq 1$:

$$\begin{aligned}\int_2^\infty \frac{1}{x(\ln x)^p}\,dx &= \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^p}\,dx \\ &= \lim_{b\to\infty} \frac{(\ln x)^{1-p}}{1-p}\Big|_2^b \\ &= \lim_{b\to\infty} \left[\frac{(\ln b)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p}\right]\end{aligned}$$

We know that $\lim_{b\to\infty} (\ln b)$ approaches ∞. Thus we need $1-p < 0$ in order for our limit to converge.

- If $p = 1$:

$$\begin{aligned}\int_2^\infty \frac{1}{x\ln x}\,dx &= \lim_{b\to\infty} \int_2^b \frac{1}{x\ln x}\,dx \\ &= \lim_{b\to\infty} \ln(\ln x)\Big|_2^b \\ &= \lim_{b\to\infty} [\ln(\ln b) - \ln(\ln 2)] \\ &\text{diverges to } \infty\end{aligned}$$

Therefore we get that the integral converges for $p > 1$.

66.

$$\begin{aligned}\int_0^\infty \frac{2x}{x^2+1}\,dx &= \lim_{b\to\infty}\int_0^b \frac{2x}{x^2+1}\,dx\\ &= \lim_{b\to\infty}\ \ln\left(x^2+1\right)\Big|_0^b\\ &= \lim_{b\to\infty}\ \ln(b^2+1)\\ &\text{diverges to } \infty\end{aligned}$$

$$\Rightarrow \int_{-\infty}^\infty \frac{2x}{x^2+1}\,dx = \int_{-\infty}^0 \frac{2x}{x^2+1}\,dx + \int_0^\infty \frac{2x}{x^2+1}\,dx \quad \text{diverges.}$$

$$\begin{aligned}\lim_{b\to\infty}\int_{-b}^b \frac{2x}{x^2+1}\,dx &= \lim_{b\to\infty}\ \ln(x^2+1)\Big|_{-b}^b\\ &= \lim_{b\to\infty}\ \left[\ln\left((-b)^2+1\right)-\ln\left(b^2+1\right)\right]\\ &= \lim_{b\to\infty}\ 0\\ &= 0\end{aligned}$$

74. a.

$$\begin{aligned}\int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx &= \lim_{b\to\infty}\int_1^b \pi\left(\frac{1}{x}\right)^2 dx\\ &= \lim_{b\to\infty}\ -\frac{\pi}{x}\Big|_1^b\\ &= \lim_{b\to\infty}\ \left(-\frac{\pi}{b}+\frac{\pi}{1}\right)\\ &= \pi\end{aligned}$$