Homework supplement about money

1 Background: Continuous Compounding

Let $A(t)$ be the balance of a bank account at time t. Suppose that interest is compounded continuously with an annual interest rate r. This means that the differential equation

$$\frac{dA}{dt} = rA \tag{1}$$

is satisfied. Note that equation (1) is the (hopefully by now familiar) differential equation for exponential growth. Recall that to solve this, we divide both sides of the equation by A and then integrate with respect to t:

$$\begin{aligned}
\frac{\frac{dA}{dt}}{A} &= r \\
\int \frac{\frac{dA}{dt}}{A}\,dt &= \int r dt \\
\int \frac{dA}{A} &= \int r dt \quad \text{(since } dA = \tfrac{dA}{dt}dt) \\
\ln A &= rt + C \\
A &= e^{rt+C} \\
A &= A_0 e^{rt}, \qquad (2)
\end{aligned}$$

where we write A_0 for the constant $e^C = A(0)$. (In class we derived equation (2) in another way: first we found the formula for the account balance when the interest is computed n times per year and then took the limit as $n \to \infty$; see page 511 of the text.) Note that equation (2) is the usual formula for exponential growth. In the next section we explore some variations that lead to slightly different formulas.

2 Some Variations

1. A bank account has an annual rate of return 0.05. You invest money in the account at a rate of \$1000 per year. If the initial balance is \$1000, when will the balance reach \$20,000? (Assume that investments are made continuously and that the return is compounded continuously.)

   Solution. Let $A(t)$ be the account balance at time t, in thousands of dollars. Then A satisfies the differential equation

$$\frac{dA}{dt} = rA + 1 \tag{3}$$

(The second term on the right-hand side is 1 because you are investing at a rate of $\$1,000$ per year and the units of A are thousands of dollars.) To solve (4), divide both sides of the equation by $rA+1$ and then integrate:

$$\begin{aligned}
\frac{\frac{dA}{dt}}{rA+1} &= 1 \\
\int \frac{\frac{dA}{dt}}{rA+1}\,dt &= \int dt \\
\int \frac{dA}{rA+1} &= \int dt \quad (\text{since } dA = \tfrac{dA}{dt}dt) \\
\frac{1}{r}\ln(rA+1) &= t+C \\
\ln(rA+1) &= rt+rC \\
rA+1 &= e^{rt+rC} = Be^{rt},
\end{aligned}$$

where we write B for the constant e^{rC}. Solving for A gives

$$A = \frac{1}{r}(Be^{rt}-1).$$

Next, we solve for B using the initial condition $A(0)=1$:

$$\begin{aligned}
1 &= \frac{1}{r}(Be^{r\cdot 0}-1) \\
1 &= 20(B-1) \\
\Rightarrow B &= \frac{21}{20}.
\end{aligned}$$

Thus

$$A(t) = 20(\tfrac{21}{20}e^{0.05t}-1).$$

Next we find the value of t such that $A(t)=20$:

$$\begin{aligned}
20 &= 20(\tfrac{21}{20}e^{0.05t}-1) \\
1 &= \tfrac{21}{20}e^{0.05t}-1 \\
2 &= \tfrac{21}{20}e^{0.05t} \\
40 &= 21e^{0.05t} \\
\ln(40) &= \ln(21)+0.05t \\
t &= 20\Big(\ln(40)-\ln(21)\Big) \text{ years.}
\end{aligned}$$

2. A bank account has an annual rate of return $r=0.10$. You invest at an annual rate of $1/A$ dollars per year when the account balance is A. Find the balance after 10 years if the initial balance is $\$1$. (Assume that investments are made continuously and that the return is compounded continuously.)

Solution. Let $A(t)$ be the account balance at time t. Then A satisfies the differential equation

$$\frac{dA}{dt} = rA + 1/A \tag{4}$$

To solve (4), divide both sides of the equation by $rA + 1/y$ and then integrate:

$$\begin{aligned}
\frac{\frac{dA}{dt}}{rA + 1/A} &= 1 \\
\int \frac{\frac{dA}{dt}}{rA + 1/A}\, dt &= \int dt \\
\int \frac{dA}{rA + 1/A} &= \int dt \quad (\text{since } dA = \tfrac{dA}{dt} dt) \\
\int \frac{A}{rA^2 + 1}\, dA &= t + C \\
\frac{1}{2r} \ln(rA^2 + 1) &= t + C \\
\ln(rA^2 + 1) &= 2rt + 2rC \\
rA^2 + 1 &= e^{2rt+2rC} = Be^{2rt}
\end{aligned}$$

where we write B for the constant e^{2rC}. Solving for A gives

$$A = \sqrt{\frac{1}{r}(Be^{2rt} - 1)}. \tag{5}$$

Next, we solve for B using the initial condition $A(0) = 1$:

$$\begin{aligned}
1 &= \sqrt{\frac{1}{r}(Be^{2r\cdot 0} - 1)} \\
1 &= 10(B - 1) \\
\Rightarrow B &= \frac{11}{10}.
\end{aligned}$$

Subsituting $B = 11/10$ and $r = 1/10$ into equation (5) gives

$$\begin{aligned}
A(t) &= \sqrt{10(\frac{11}{10}e^{t/5} - 1)} \\
&= \sqrt{11e^{t/5} - 10}.
\end{aligned}$$

So

$$\begin{aligned}
A(10) &= \sqrt{11e^{10/5} - 10} \\
&= \sqrt{11e^{2} - 10} \text{ dollars}
\end{aligned}$$

3.

4. A retirement account has an annual rate of return 10%. The broker charges fees at an annual rate of 5%. If the initial balance is \$1,0000, how much is collected in fees after 10 years? (Assume that the return and fees are computed continuously.)

Solution. Let $r = 0.10$ and $s = 0.05$. Let $A(t)$ be the account balance at time t, in thousands of dollars. Then A satisfies the differential equation

$$\frac{dA}{dt} = rA - sA = (r - s)A \tag{6}$$

Equation (6) is the differential equation for exponential growth with rate $r - s$. With the initial condition $A(0) = 1$, the solution is

$$A(t) = e^{(r-s)t} .$$

The fees df collected between t and $t + dt$ is $sA(t)dt$, so the total fees collected is

$$\begin{aligned} \int df &= \int_0^{10} sA(t)\, dt \\ &= \int_0^{10} 0.05e^{0.05t}\, dt \\ &= e^{0.05t}\Big]_0^{10} \\ &= \sqrt{e} - 1 \text{ thousand dollars.} \end{aligned}$$