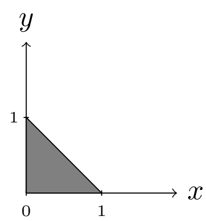
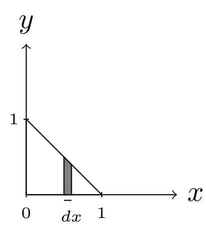
Mass Problems

1. Find the mass of the triangular region below. All lengths are in meters, and the density of the region is given by $\delta(x) = x$ grams/m^2. (Note that since the object is 2-dimensional, its density is its mass per unit area.)



Solution. Since the density is a function of x, we divide the region into thin vertical strips of thickness dx as shown in the following figure:



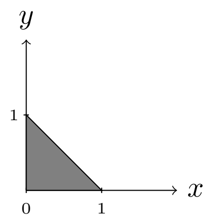
(Question: What kind of strips would we use if the density were a function of y?) Since the height of the strip is $1 - x$, its area is $(1 - x)dx$, and hence its mass is

$$\begin{aligned} dm &= (\text{density of strip}) \times (\text{area of strip}) \\ &= x(1-x)dx \end{aligned}$$

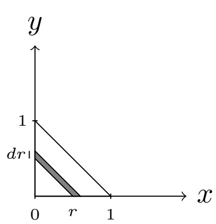
Hence the total mass of the triangle is

$$\begin{aligned} m &= \int dm \\ &= \int_0^1 x(1-x)\, dx \\ &= \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3]_0^1 = \tfrac{1}{6} \text{ grams} \end{aligned}$$

2. Find the mass of the triangular region below. All lengths are in meters, and the density of the region is given by $\delta(x, y) = e^{(x+y)^2}$ grams/m^2.



Solution. Note that we can write the density as $\delta(r) = e^{r^2}$, where $r = x + y$. For r in $[0, 1]$, the graph of $x + y = r$ is a diagonal line that passes through the point $(r, 0)$. This suggests dividing the triangle into thin diagonal strips as shown in the following figure:



In the picture above, the strip is bounded by coordinate axes and the lines $x + y = r$ and $x + y = r + dr$. Its area is rdr, which can be seen from the following calculation: Let $A(r) = \frac{1}{2}r^2$ be the area of a triangle whose base and height are both r, and let $dA = A(r + dr) - A(r)$. Then

$$\begin{aligned}\text{area of strip} &= dA \\ &= rdr,\end{aligned}$$

where the last line follows from differentiation. It follows that the mass of the strip is

$$\begin{aligned}dm &= (\text{density of strip}) \times (\text{area of strip}) \\ &= e^{r^2} rdr\end{aligned}$$

Hence the total mass of the triangle is

$$\begin{aligned}m &= \int dm \\ &= \int_0^1 re^{r^2} dr \\ &= \tfrac{1}{2}e^{r^2}\Big]_0^1 = \tfrac{1}{2}(e - 1) \ \text{ grams}\end{aligned}$$

3. A thin plate occupies the region of the plane bounded by the circle $x^2 + y^2 = 1$. Find the total mass if the density at the point (x, y) is given by $\delta(x, y) = \frac{1}{\sqrt{x^2+y^2}}$. (Hint: divide the region into thin circular "rings" centered at the origin.)

   Answer: 2π.

4. The region bounded by the graph of $y = x^2$ and the x-axis, between 0 and 1, is revolved about the x-axis. The resulting solid has density given by $\delta(x) = x$. (Here the object is 3-dimensional, so its density is its mass per unit volume.) Find the total mass.

   Answer: $\pi/6$.