1. (20 points.) The region bounded by the graphs of \( y = 2\sqrt{x} \), \( y = 2 \) and \( x = 0 \) is revolved about the \( x \)-axis. Find the volume of the resulting solid.

**Solution.** Find intersection point:

\[
\begin{align*}
2 &= 2\sqrt{x} \\
\sqrt{x} &= 1 \\
x &= 1
\end{align*}
\]

Washer:

\[
V = \pi \int_0^1 2^2 - (2\sqrt{x})^2 \, dx
= \pi \int_0^1 4 - 4x \, dx
= \pi \left[ 4x - 2x^2 \right]_0^1 = 2\pi.
\]

2. (20 points.) The region bounded by the graphs of \( y = \sqrt{x} \) and \( y = x^2/8 \) is revolved about the \( y \)-axis. Find the volume of the resulting solid.

**Solution.** Find intersection points:

\[
\begin{align*}
\sqrt{x} &= x^2/8 \\
8\sqrt{x} &= x^2 \\
64x &= x^4 \\
0 &= x^4 - 64x = x(x^3 - 64)
\end{align*}
\]

\( \Rightarrow x = 0, 4 \)

Shell:

\[
V = 2\pi \int_0^4 x(\sqrt{x} - x^2/8) \, dx
= 2\pi \int_0^4 x^{3/2} - x^3/8 \, dx
= 2\pi \left[ \frac{2}{3}x^{5/2} - \frac{1}{32}x^4 \right]_0^4
= 2\pi \left[ \frac{2}{3} \cdot 32 - 8 \right] = \frac{48\pi}{5}
\]

3. (20 points.) Find the area of the region bounded by the graphs of \( y = 0 \), \( y = 2 \), \( y = \sqrt{x} \) and \( y = \sqrt{x-1} \).
Solution. If you draw a picture you see that horizontal strips are better. The curve on the left is $x = y^2$ and the curve on the right is $x = y^2 + 1$. So

$$A = \int_0^2 (y^2 + 1) - y^2 \, dy$$
$$= \int_0^2 1 \, dy = 2$$

4. (20 points.) A car traveling at a constant speed of 10 miles per hour has a deteriorating engine. After $t$ hours the gas mileage is $20/(t + 1)$ miles per gallon. How far will the car go on 6 gallons of gas?

**Solution.** Let $s(t)$ be the position at time $t$. Then

$$\frac{ds}{dt} = 10; \quad \frac{ds}{dg} = \frac{20}{t + 1}.$$  

Re-arranging terms gives

$$dg = \frac{1}{2} (t+1) dt$$

The gas used by time $T$ is

$$\int dg = \int_0^T \frac{1}{2} (t+1) \, dt$$
$$= \left. \frac{1}{4} (t+1)^2 \right|_0^T$$
$$= \frac{1}{4} \left[ (T+1)^2 - 1 \right]$$

This is 6 when $T = 4$. After 4 hours the car has gone $10 \times 4 = 40$ miles.

5. (20 points.) Find the mass of the triangular region below. All lengths are in meters, and the density of the region is given by $\delta(y) = y$ grams/m$^2$.

**Solution.** Consider a horizontal strip of width $dy$ at height $y$. It has length $L(y) = 2(1 - y)$, which can be seen by the following calculation: The line in the first quadrant is the graph of $y = 1 - x$, or $x = 1 - y$, and $L(y) = 2x = 2(1 - y)$. Thus

$$\text{strip area} = L(y)dy$$
$$\Rightarrow \text{strip mass} \; dm = \delta(y)L(y)dy$$
$$= y \cdot 2(1 - y)dy$$
Hence

\[ m = \int dm \]
\[ = \int_0^1 2y(1 - y) \, dy \]
\[ = 2 \int_0^1 y - y^2 \, dy \]
\[ = 2 \left[ \frac{1}{2} y^2 - \frac{1}{3} y^3 \right]_0^1 = \frac{1}{3} \text{ grams} \]

6. (20 points.) Find the area of the surface obtained by revolving the curve \( y = x + 1, \ 0 \leq x \leq 1 \) about the \( y \)-axis.

**Solution.** We have

\[ ds = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]
\[ = \sqrt{1 + 1^2} \, dx \]
\[ = \sqrt{2} \, dx, \]

so

\[ A = \int_0^1 2\pi x \, ds \]
\[ = \int_0^1 2\pi x \sqrt{2} \, dx \]
\[ = 2\sqrt{2} \pi \int_0^1 x \, dx \]
\[ = 2\sqrt{2} \pi \left[ \frac{1}{2} x^2 \right]_0^1 = \sqrt{2} \pi. \]

7. (20 points.) A rope 50 meters long weighing 2 Newtons per meter is hanging over the edge of a tall building and does not touch the ground. How much work is required to lift the entire rope to the top of the building?

**Solution.** When the rope extends a distance \( x \) from the top of the building, the weight of the part hanging down is \( F(x) = 2x \). Thus the work is

\[ W = \int_0^{50} F(x) \, dx \]
\[ = \int_0^{50} 2x \, dx \]
\[ = x^2 \bigg|_0^{50} = 2500 \text{ Nm} \]