Math 125A **Additional Problems** **Winter 2011**

1. Prove that for all sets A, B and C we have

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

2. Prove that there exists no smallest positive real number. Does there exist a smallest positive rational number? Given a real number x, does there exist a smallest real number $y > x$?

3. Prove that

$$[a, b] = \{y \in \mathbf{R} : \text{there exist } s, t \in [0, 1] \text{ such that } s + t = 1 \text{ and } y = sa + tb\} .$$

4. Let $a_1, a_2, \ldots$ be a bounded sequence of real numbers. Define

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_n, a_{n+1}, \ldots\}. \tag{1}$$

   (a) Show that the limsup is defined (i.e., show that the limit on the right-hand side of the equation exists).

   (b) Show that this definition of limsup is equivalent to the definition given in problem 18 on page 62 of Rosenlicht.

   (c) When is it true that

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

$$\limsup_{n\to\infty} c\, a_n = c \limsup_{n\to\infty} a_n \quad ?$$

5. Find a subset of $\mathbf{R}$ that is neither open nor closed.

6. Prove that the set $\{(x, 0) : x \in \mathbf{R}\}$ is not an open subset of $\mathbf{R}^2$.

7. Find all subsets of $\mathbf{R}$ that are both open and closed.

8. Find an interval $[a, b]$ in $\mathbf{R}$ such that the intersection $[a, b] \cap \mathbf{Q}$ is both an open and closed subset of the metric space $\mathbf{Q}$.

9. Let E be a metric space and S a non-empty subset of E. For points $p \in E$, define the **distance** from p to S by

$$d(p, S) = \inf\{d(p, s) : s \in S\}.$$

   Show that the set $\{y : d(y, S) < 1\}$ is open.

10. Construct a set with exactly three cluster points.